Introduction into Partial Differential Equations

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Chapter 0

Errata

By vote of the class, the original version of the script will remain online and unchanged for the semester. Instead I will make any changes to this version. Any serious changes I will list here. Typos and small changes I will not.

Last change: December 12, 2023

- Moved the definition of 'embedded' from Definition 2.5 to Definition 2.3. Be aware: this is not the usual definition of an embedded submanifold and not equivalent to the usual definition.
- Corrected the definition of λ in the text preceding Theorem 2.7 and in its proof.
- Labelled part (a) and part (b) in Theorem 3.6. Part (b) is now an equivalence.
- I added some steps in the calculation surrounding the fundamental solution and Green's function to make it clearer. In particular that the signs are in fact correct in Theorem 3.2 and the calculation leading up to Green's representation formula.
- I changed the way that the Green's function on $B(a, r)$ is deduced from the one on the unit ball. This enabled a simplification of Poisson's representation formula to just prove the unit ball case. I also added a full proof of the final part after the three observations.
- Corrected signs in Lemma 4.3.
- Changed the notation for inversion in the sphere from a tilde to ι.
- Missing scaling factor in the heat kernel of an interval.
- Added Weierstrass' counterexample to Dirichlet principle and limit of weak solutions of the wave equation.

Chapter 1

First Order PDEs

In this introductory chapter we first introduce partial differential equations and then consider first order partial differential equations. We shall see that they are simpler than higher order partial differential equations. In contrast to higher order partial differential equations these first order partial differential equations are similar to ordinary differential equations and can be solved by using the theory of ordinary differential equations. After this introductory chapter we shall focus on second order partial differential equations. Before we consider the three main examples of second order differential equations we introduce some general concepts in the next chapter. These general concepts are partially motivated by observations contained in the first chapter.

A partial differential equation is an equation on the partial derivatives of a function depending on at least two variables.

Definition 1.1. *A possibly vector valued equation of the following form*

$$F\left(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0$$

is called partial differential equation of order k. Here F is a given function and u an unknown function. The expressions $D^k u$ denotes the vector of all partial derivatives of the function u of order k. The function u is called a solution of the differential equation, if u is k times differentiable and obeys the partial differential equation.

On open subsets $\Omega \subset \mathbb{R}^n$ we denote the partial derivatives of higher order by $\partial^\gamma = \prod_i \partial_i^{\gamma_i} = \prod_i (\frac{\partial}{\partial x_i})^{\gamma_i}$ with multi-indices $\gamma \in \mathbb{N}_0^n$ of length $|\gamma| = \sum_i \gamma_i$. The multi-indices are ordered by $\delta \leq \gamma \iff \delta_i \leq \gamma_i$ for $i = 1, \ldots, n$. The partial derivative acts only on the immediately following function; they only act on a product of functions if the product is grouped together in brackets.

1.1 Homogeneous Transport Equation

One of the simplest partial differential equations is the transport equation:

$$\dot{u} + b \cdot \nabla u = 0.$$

Here $\dot{u}$ denotes the partial derivative $\frac{\partial u}{\partial t}$ of the unknown function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $b \in \mathbb{R}^n$ is a vector, and the product $b \cdot \nabla u$ denotes the scalar product of the vector b with the vector of the first partial derivatives of u with respect to x:

$$b \cdot \nabla u(x,t) = b_1 \frac{\partial u(x,t)}{\partial x_1} + \ldots + b_n \frac{\partial u(x,t)}{\partial x_n}.$$

Let us first assume that $u(x,t)$ is a differentiable solution of the transport equation. For all fixed $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ the function

$$z(s) = u(x_0 + s \cdot b, t_0 + s)$$

is a differentiable function on $s \in \mathbb{R}$, whose first derivative vanishes:

$$z'(s) = b\nabla u(x_0 + s \cdot b, t_0 + s) + \dot{u}(x_0 + s \cdot b, t_0 + s) = 0.$$

Therefore u is constant along all parallel straight lines in direction of $(b,1)$. Furthermore, u is completely determined by the values on all these parallel straight lines.

Initial Value Problem 1.2. *We seek a solution $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ of the transport equation $\dot{u} + b \cdot \nabla u = 0$ with given $b \in \mathbb{R}^n$, which at $t = 0$ is equal to some given function $g : \mathbb{R}^n \to \mathbb{R}$. We call this the Cauchy problem (or initial value problem) for the transport equation.*

With the additional initial data, we can now uniquely determine a solution. All parallel straight lines in direction of $(b,1)$ intersect $\mathbb{R}^n \times \{0\}$ exactly once:

$$(x_0 + sb, t_0 + s) \in \mathbb{R}^n \times \{0\} \iff s = -t_0.$$

Thus the value of u on each straight line is determined by the initial condition. These lines are in general called characteristic curves. The solution has to be equal to $u(x,t) = u(x - tb, 0) = g(x - tb)$. If g is differentiable on $\mathbb{R}^n$, then this function indeed solves the transport equation. In this case the initial value problem has a unique solution. Otherwise, if g is not differentiable on $\mathbb{R}^n$, then the initial value problem does not have a solution. As we have seen above, whenever the initial value problem has a solution, then the function $u(x,t) = g(x - bt)$ is the unique solution. So it might be that this candidate is a solution in a more general sense.

1.2 Inhomogeneous Transport Equation

Now we consider the corresponding inhomogeneous transport equation:

$$\dot{u} + b \cdot \nabla u = f.$$

Again $b \in \mathbb{R}^n$ is a given vector, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a given function and $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the unknown function.

Initial Value Problem 1.3. *Given a vector $b \in \mathbb{R}$, a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and an initial value $g : \mathbb{R}^n \to \mathbb{R}$, we seek a solution to the Cauchy problem for the inhomogeneous transport equation: a function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ that satisfies*

$$\dot{u} + b \cdot \nabla u = f \qquad \textit{with} \qquad u(x,0) = g(x).$$

Similar to the homogeneous case, we define for each $(x_0, 0) \in \mathbb{R}^n \times \mathbb{R}$ the function $z(s) = u(x_0 + sb, s)$ which solves

$$z'(s) = b \cdot \nabla u(x_0 + sb, s) + \dot{u}(x_0 + sb, s) = f(x_0 + sb, s).$$

Notice that the right hand side is only a function of s. Moreover $z(0) = u(x_0, 0) = g(x_0)$ is known. Thus we can integrate and determine $z(s)$ completely. This tells us the value of u and any point on the line $(x_0 + sb, s) \in \mathbb{R}^n \times \mathbb{R}$.

We can also gather this information into a formula for u. The point (x, t) lies on the line $(x_0 + sb, s)$ with $s = t$ and $x_0 = x - tb$. Therefore

$$\begin{aligned} u(x,t) = z(t) = z(0) + \int_0^t z'(s)\ \mathrm{d}s &= g(x_0) + \int_0^t f(x_0 + sb, s)\ \mathrm{d}s \\ = g(x - tb) + \int_0^t f(x + (s-t)b, s)\ \mathrm{d}s. \end{aligned}$$

We observe that this formula is analogous to the formula for solutions of inhomogeneous initial value problems of linear ODEs. The unique solution is the sum of the unique solution of the corresponding homogeneous initial value problem and the integral over solution of the homogeneous equation with the inhomogeneity as initial values. We obtained these solutions of the first order homogeneous and inhomogeneous transport equation by solving an ODE. We shall generalise this method in Section 1.5 and solve more general first order PDEs by solving an appropriate chosen system of first order ODEs.

1.3 Scalar Conservation Laws

In this section we consider the following class of non-linear first order differential equations

$$\dot{u}(x,t) + \frac{\partial f(u(x,t))}{\partial x} = \dot{u}(x,t) + f'(u(x,t)) \cdot \frac{\partial u(x,t)}{\partial x} = 0$$

for a smooth function $f : \mathbb{R} \to \mathbb{R}$. Here $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the unknown function. This equation is called a scalar conservation law and is a non-linear first order PDE. For any compact interval $[a, b]$ we calculate

$$\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b \dot{u}(x,t)dx = -\int_a^b \frac{\partial f(u(x,t))}{\partial x}dx = f(u(a,t)) - f(u(b,t)).$$

This is the meaning of a conservation law: the change of the integral of $u(\cdot, t)$ over $[a, b]$ is equal to the 'flux' of $f(u(x, t))$ through the 'boundary' $\partial[a, b] = \{a, b\}$.

Thinking of t as time, the natural boundary condition to consider is $u(x, 0) = g(x)$ for all $x \in \mathbb{R}$ with some given function $g : \mathbb{R} \to \mathbb{R}$. Let us try to apply the method of characteristics to these equations, namely we assume that there exists a solution u try to understand how the value of u changes along a curve $(x(s), s)$ in its domain. The difference to the transport equation is that we do not assume that the curves are straight lines; it remains to be seen which curves we should choose. Let $z(s) = u(x(s), s)$. The derivative is

$$z'(s) = \frac{\partial u(x(s), s)}{\partial x}x'(s) + \dot{u}(x(s), s)$$

Hence if we choose the curve $x(s)$ with the property that $x'(s) = f'(u(x(s), s))$ then

$$z'(s) = \frac{\partial u(x(s), s)}{\partial x}x'(s) + \dot{u}(x(s), s) = \frac{\partial u(x(s), s)}{\partial x}f'(u(x(s), s)) + \dot{u}(x(s), s) = 0.$$

This shows that z is constant along these particular curves.

There remain two things to determine: what is the value of z and does there even exist a curve $x(s)$ with the required property? We make the assumption that the characteristic curve begins at the point $(x_0, 0)$. In other words $x(0) = x_0$. By the constancy of z and the initial conditions we have $z(s) = u(x(0), 0) = u(x_0, 0) = g(x_0)$. This answers the first question. The second question is now answerable too: the derivative of $x(s)$ is constant equal to

$$x'(s) = f'(u(x(s), s)) = f'(z(s)) = f'(g(x_0)).$$

The characteristic curve is therefore $x(s) = x_0 + sf'(g(x_0))$. Together this shows that the solution of the PDE is uniquely determine from the initial condition, if it exists.

Instead of thinking about a single characteristic curve and initial point, let us think about all characteristic curves. This point of view implies the solution obeys

$$u(x + tf'(g(x)), t) = g(x) \quad \text{for all} \quad (x, t) \in \mathbb{R}^2.$$

The characteristic curves with initial points $x_1, x_2 \in \mathbb{R}^n$ with $g(x_1) \neq g(x_2)$ might intersect at $t \in \mathbb{R}^+$. In this case the method of characteristic implies $g(x_1) = u(x_1 + tf'(g(x_1)), t) = u(x_2 + tf'(g(x_2)), t) = g(x_2)$, which is impossible. This situation is called crossing characteristics. But otherwise the above implicit equation for u can be solved and defines a solution to the PDE.

Theorem 1.4. *If $f \in C^2(\mathbb{R}, \mathbb{R})$ and $g \in C^1(\mathbb{R}, \mathbb{R})$ with $f''(g(x))g'(x) > -\alpha$ for all $x \in \mathbb{R}$ and some $\alpha \geq 0$, then there is a unique C^1-solution of the initial value problem for the scalar conservation law*

$$\frac{\partial u(x,t)}{\partial t} + f'(u(x,t))\frac{\partial u(x,t)}{\partial x} = 0 \quad \text{with} \quad u(x,0) = g(x)$$

on $(x,t) \in \mathbb{R} \times [0, \alpha^{-1})$ for $\alpha > 0$ and on $(x,t) \in \mathbb{R} \times [0, \infty)$ for $\alpha = 0$.

Proof. By the method of characteristic the solution $u(x,t)$ is on the lines $x + tf'(g(x))$ equal to $g(x)$. For all $t \geq 0$ with $1 - t\alpha > 0$ the derivative of $x \mapsto x + tf'(g(x))$ obeys

$$1 + tf''(g(x))g'(x) \geq 1 - t\alpha > 0.$$

This implies $\lim_{x \to \pm\infty} x + tf'(g(x)) = \pm\infty$. So $x \mapsto x + tf'(g(x))$ is a C^1-diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Therefore there exists for any $y \in \mathbb{R}$ a unique x with $x + tf'(g(x)) = y$. Then $u(y,t) = g(x)$ solves the initial value problem. □

Example 1.5. *For $n = 1$ and $f(u) = \frac{1}{2}u^2$ we obtain Burgers equation:*

$$\dot{u}(x,t) + u(x,t)\frac{\partial u(x,t)}{\partial x} = 0.$$

The solutions of the corresponding characteristic equations are $x(t) = x_0 + g(x_0)t$. Therefore the solutions of the corresponding initial value problem obey

$$u(x + tg(x), t) = g(x).$$

If g is continuously differentiable and monotonic increasing, then for all $t \in [0, \infty)$ the map $x \mapsto x + tg(x)$ is a C^1-diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and there is a unique C^1-solution on $\mathbb{R} \times [0, \infty)$. More generally, if $g'(x) > -\alpha$ with $\alpha \geq 0$, then there is a unique C^1-solution on $\mathbb{R} \times [0, \alpha^{-1})$ for $\alpha > 0$ and $(x,t) \in \mathbb{R} \times [0, \infty)$ for $\alpha = 0$.

1.4 Noncharacteristic Hypersurfaces

Until now we have only considered specific PDEs where one variable was labelled 'time' and the initial conditions was $t = 0$. In this section we shall consider boundary conditions for the general first order PDE:

$$F(\nabla u(x), u(x), x) = 0$$

on the domain $\Omega \subseteq \mathbb{R}^n$ with the boundary condition $u(y) = g(y)$ for all $y \in \Sigma$. Here u is a real unknown function on an open domain $\Omega \subset \mathbb{R}^n$ and F is a real function on an open subset of $W \subset \mathbb{R}^n \times \mathbb{R} \times \Omega$. For the boundary condition we assume that $\Sigma = \{x \in \Omega \mid \varphi(x) = \varphi(x_0)\}$ is the level-set of the function φ, which we call a hypersurface.

We will first show that locally every Cauchy problem can be brought into the following form:

$$u(y) = g(y) \text{ for all } y \in \Omega \cap H \text{ with } H = \{x \in \mathbb{R}^n \mid x \cdot e_n = x_0 \cdot e_n\}.$$

Here $e_n = (0, \dots, 0, 1)$ denotes the n-th element of the canonical basis and H the unique hyperplane through $x_0 \in \Omega$ orthogonal to e_n. If $\nabla\varphi(x_0) \neq 0$ we may assume without loss of generality that $\frac{\partial\varphi}{\partial x_n}(x_0) \neq 0$ (relabel the variables if necessary). Then we apply the inverse function theorem to $x \mapsto \Phi(x) = (x_1, \dots, x_{n-1}, \varphi(x))$ to get a continuously differentiable coordinate transformation $x = \Phi^{-1}(y)$ in a neighbourhood of x_0. This coordinate change has the property that $\varphi(x) = \varphi(x_0)$ if and only if $y \cdot e_n = y_n = \varphi(x_0)$. We say that the boundary has been straighten at x_0. Then by the chain rule the composition $u = v \circ \Phi$ of a function $v : \Omega' \to \mathbb{R}$ with Φ obeys

$$\nabla u(x) = \nabla v(\Phi(x)) \cdot \Phi'(x) = \nabla v(y) \cdot \Phi'\left(\Phi^{-1}(y)\right).$$

Here ∇v and ∇u are row vectors and $\Phi'(x)$ the Jacobi matrix. Hence u solves the PDE

$$F(\nabla u(x), u(x), x) = 0$$

if and only if v solves the PDE

$$G(\nabla v(y), v(y), y) := F\left(\nabla v(y) \cdot \Phi'\left(\Phi^{-1}(y)\right), v(y), \Phi^{-1}(y)\right) = 0.$$

Thus we can indeed assume locally (the coordinate change is only guaranteed to exist in a neighbourhood of x_0) that the boundary is a hyperplane, at the cost of changing the form of the PDE.

Next we ask the question: given the values of u on the hypersurface H is there anything else we can determine about u on the hypersurface? Can we determine the value of its derivatives for example, or can we see immediately that there is no possible u (like for some situations of Burgers' equation)?

We can compute the partial derivatives in most directions at $x_0 \in H$. Observe

$$\frac{\partial u(x_0)}{\partial x_1} = \lim_{h \to 0} \frac{u(x_0 + he_1) - u(x_0)}{h} = \lim_{h \to 0} \frac{g(x_0 + he_1) - g(x_0)}{h} = \frac{\partial g(x_0)}{\partial x_1}.$$

This also works for the directions $x_2, \dots, x_{n-1}$ which lie in the hyperplane. This idea does not determine $\frac{\partial u(x_0)}{\partial x_n}$, but we have not used the PDE yet. If we substitute all the values we know, there is only one free variable in the PDE:

$$F(\nabla u(x_0), u(x_0), x_0) = F\left(\frac{\partial g(x_0)}{\partial x_1}, \dots, \frac{\partial g(x_0)}{\partial x_{n-1}}, p_n, g(x_0), x_0\right) = 0.$$

Whether or not this has a solution depends on both the PDE F and the initial condition g. However, if there does exist a solution then there is a simple criterion depending only on F that ensures that is solvable in a neighbourhood of x_0.

Definition 1.6. *Consider the PDE as a function of $2n+1$ variables $F(p,z,x)=0$ and suppose that there is a solution (p_0,z_0,x_0). The hyperplane $H=\{x_n=x_{0,n}\}$ is called noncharacteristic at x_0 if*

$$\frac{\partial F}{\partial p_n}(p_0,z_0,x_0)\neq 0.$$

To understand the name 'noncharacteristic' let us consider the example

$$\frac{\partial u}{\partial x_1}=0, \qquad u(x_1,0)=g(x_1).$$

The PDE in this case is $F(p_1,p_2,z,x_1,x_2)=p_1$, which clearly does not enjoy the non-characteristic property. We see that the initial condition is fighting against the PDE; they are only compatible if g is constant. And even if they happen to be compatible then the initial condition does not determine $\frac{\partial u}{\partial x_2}$ on $H=\{x_2=0\}$. If we apply the method of characteristics to this PDE, we must try to find a curve $(x_1(s),x_2(s))$ along which $z(s)=u(x_1(s),x_2(s))$ is nicely behaved. Differentiating z gives

$$z'=\frac{\partial u}{\partial x_1}x_1'+\frac{\partial u}{\partial x_2}x_2',$$

which 'aligns' with the PDE if we choose $x_1'=1$ and $x_2'=0$. However this choice of characteristics gives $x_1(s)=x_{0,1}+s, x_s(s)=x_{0,2}$, which lies in the hyperplane. The method fails to be useful because no points in the domain can be reached by characteristics starting on the hyperplane.

Lemma 1.7. *Let $F:W\to\mathbb{R}$ and $g:H\to\mathbb{R}$ be continuously differentiable, $x_0\in\Omega\cap H$, $z_0=g(x_0)$ and $p_{0,1}=\frac{\partial g(x_0)}{\partial x_1},\dots,p_{0,n-1}=\frac{\partial g(x_0)}{\partial x_{n-1}}$. If there exists $p_{0,n}$ with $F(p_0,z_0,x_0)=0$ and H is noncharacteristic at x_0 then in an open neighbourhood $\Omega_{x_0}\subset\Omega$ of x_0 there exists for $x\in\Omega_{x_0}\cap H$ a unique solution q of*

$$F(q(x),g(x),x)=0, \qquad q_i(x)=\frac{\partial g(x)}{\partial x_i}\ \textit{for } i=1,\dots,n-1 \qquad \textit{and} \qquad q(x_0)=p_0.$$

Proof. Consider the function $(x,q_n)\mapsto F(q_1(x),\dots,q_{n-1}(x),q_n,g(x),x)$. This takes the value 0 at $(x_0,p_{0,n})$. The noncharacteristic assumption means that we can apply the implicit function theorem to define q_n as a unique function of x in a neighbourhood of x_0. □

1.5 Method of Characteristics

In this section continue to consider the general first order PDE and try to formalise the method of characteristics, which thus far we have developed only ad hoc. We try to obtain the solution to the PDE by understanding the function u along a curve in the domain.

For a clever choice of the curve this reduces to the solution of an appropriate system of first order ODEs. So let $x(s)$ be a curve in the domain of the PDE and $z(s) = u(x(s))$ be the value of u along the curve. The new ingredient is that we must also consider $p(s) = \nabla u(x(s))$, the gradient of u along this curve. But how should be choose the curve $s \mapsto x(s)$? For this purpose we first differentiate

$$p_i'(s) = \frac{d}{ds}\frac{\partial u(x(s))}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i} x_j'(s).$$

The total derivative of $F(\nabla u(x), u(x), x) = 0$ with respect to x_i gives

$$\begin{aligned} 0 &= \frac{dF(\nabla u(x), u(x), x)}{dx_i} = \\ &= \sum_{j=1}^{n} \frac{\partial F(\nabla u(x), u(x), x)}{\partial p_j}\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \frac{\partial F(\nabla u(x), u(x), x)}{\partial z}\frac{\partial u(x)}{\partial x_i} + \frac{\partial F(\nabla u(x), u(x), x)}{\partial x_i}. \end{aligned}$$

Due to the commutativity $\partial_i \partial_j u = \partial_j \partial_i u$ of the second partial derivatives we obtain

$$\sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}\frac{\partial^2 u(x(s))}{\partial x_j \partial x_i} + \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) + \frac{\partial F(p(s), z(s), x(s))}{\partial x_i} = 0.$$

We want to eliminate the explicit dependence on u from all our equations. If we compare this equation with the derivative of p_i we see that we should choose the vector field for the characteristic curves as

$$x_j'(s) = \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}.$$

This choice allows us to rewrite the equation above for p' as

$$\begin{aligned} p_i'(s) &= \sum_{j=1}^{n} \frac{\partial^2 u(x(s))}{\partial x_j \partial x_i}\frac{\partial F(p(s), z(s), x(s))}{\partial p_j} \\ &= -\frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) - \frac{\partial F(p(s), z(s), x(s))}{\partial x_i}. \end{aligned}$$

Finally we differentiate

$$z'(s) = \frac{d}{ds} u(x(s)) = \sum_{j=1}^{n} \frac{\partial u(x(s))}{\partial x_j} x_j'(s) = \sum_{j=1}^{h} p_j(s) \frac{\partial F(p(s), z(s), x(s))}{\partial p_j}.$$

In this way we indeed obtain the following system of first order ODEs:

$$\begin{aligned} x_i'(s) &= \frac{\partial F(p(s), z(s), x(s))}{\partial p_i} \\ p_i'(s) &= -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) \\ z'(s) &= \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s). \end{aligned}$$

This is a system of first order ODEs with $2n+1$ unknown real functions. Importantly this is a 'closed' system; it only depends on these $2n+1$ functions, not on any other information from u. This is a little surprising, particularly that p', which is effectively a certain second derivative of u, only depends on the location x, the value z, and the first derivatives p. The fact that this idea of characteristics leads to a finite system of ODEs is what makes this an effective method. Let us summarise these calculations in the following theorem:

Theorem 1.8. *Let F be a real differentiable function on an open subset $W \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ a twice differentiable solution on an open subset $\Omega \subset \mathbb{R}^n$ of the first order PDE $F(\nabla u(x), u(x), x) = 0$. For every solution $s \mapsto x(s)$ of the ODE*

$$x_i'(s) = \frac{\partial F}{\partial p_i}(\nabla u(x(s)), u(x(s)), x(s))$$

the functions $p(s) = \nabla u(x(s))$ and $z(s) = u(x(s))$ solve the ODEs

$$p_i'(s) = -\frac{\partial F(p(s), z(s), x(s))}{\partial x_i} - \frac{\partial F(p(s), z(s), x(s))}{\partial z} p_i(s) \text{ and}$$
$$z'(s) = \sum_{j=1}^{n} \frac{\partial F(p(s), z(s), x(s))}{\partial p_j} p_j(s).$$

□

This theorem can be used to address the uniqueness of the solution of PDE, reducing it to the question of uniqueness of the solution of this system of ODEs. This is useful because we have many theorems that tell us when a system of ODEs is unique. For example, the Picard-Lindelöf theorem tells us the solution is uniquely determined by initial conditions if the right hand side is Lipschitz.

We must also pay attention to the logical structure of this theorem. It says *if a solution to the PDE exists* then it solves the ODE; it tells us where to look for potential solutions. But that was not the task we set for ourselves at the outset of this section. We want to prove that a solution of the PDE does in fact exist. We have seen that global solutions may not exist due to crossing characteristics, so the best we can hope for is a local existence result. This takes a little work but is achieved in the following theorem.

Theorem 1.9. *Let $F : W \to \mathbb{R}$ and $g : H \to \mathbb{R}$ be three times differentiable functions. Suppose we have a point $(p_0, z_0, x_0) \in W$ with*

$$F(p_0, z_0, x_0) = 0, \qquad z_0 = g(x_0), \qquad p_{0,1} = \tfrac{\partial g(x_0)}{\partial x_1}, \ \ldots, \ p_{0,n-1} = \tfrac{\partial g(x_0)}{\partial x_{n-1}}.$$

Furthermore, assume that H is noncharacteristic at x_0. Then in a neighbourhood $\Omega_{x_0} \subset \Omega$ of x_0 there exists a unique solution of the boundary value problem

$$F(\nabla u(x), u(x), x) = 0 \quad \text{for} \quad x \in \Omega_{x_0} \quad \text{and} \quad u(y) = g(y) \quad \text{for} \quad y \in \Omega_{x_0} \cap H.$$

Proof. The strategy of this proof is to solve the system of ODEs given by the method of characteristics and show that it does solve the PDE and the initial conditions. First we need to translate the initial conditions of the PDE to initial conditions for the ODEs. By Lemma 1.7 there exists a solution q on an open neighbourhood of x_0 in H of the following equations

$$F(q(y), g(y), y) = 0, \qquad q_i(y) = \tfrac{\partial g(y)}{\partial x_i} \text{ for } i = 1, \ldots, n-1 \quad \text{and} \qquad q(x_0) = p_0.$$

If F is twice and g are three times differentiable then the implicit function theorem yields a twice differentiable solution. The Picard-Lindelöf theorem shows that the following initial value problem has for all y in the intersection of an open neighbourhood of x_0 with H a unique solution:

$$\begin{aligned}
x_i'(s) &= \frac{\partial F}{\partial p_i}(p(s), z(s), x(s)) && \text{with} \quad x(0) = y \\
p_i'(s) &= -\frac{\partial F}{\partial x_i}(p(s), z(s), x(s)) - \frac{\partial F}{\partial z}(p(s), z(s), x(s))p_i(s) && \text{with} \quad p(0) = q(y) \\
z'(s) &= \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(p(s), z(s), x(s))p_j(s) && \text{with} \quad z(0) = g(y).
\end{aligned}$$

We denote the family of solutions by $(x(y, s), p(y, s), z(y, s))$. For a neighbourhood $\Omega_{x_0} \ni x_0$ there exists an $\epsilon > 0$ such that these solutions are uniquely defined on $(y, s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$. This is a local proof so let us just write Ω instead of Ω_{x_0}. Since F and g are three times differentiable all coefficients and initial values are twice differentiable. The theorem on the dependence of solutions of ODEs on the initial values gives that $(y, s) \mapsto (x(y, s), p(y, s), z(y, s))$ is on $(\Omega \cap H) \times (-\epsilon, \epsilon)$ twice differentiable.

Now let us examine the characteristic curves in more detail. The function $(y, s) \mapsto x(y, s)$ on $(\Omega \cap H) \times (-\epsilon, \epsilon) \to \mathbb{R}^n$ has at $(y, s) = (x_0, 0)$ the Jacobi matrix

$$\begin{pmatrix}
1 & 0 & \ldots & 0 & \frac{\partial F(p_0, z_0, x_0)}{\partial p_1} \\
 & & \vdots & & \vdots \\
0 & 0 & \ldots & 1 & \frac{\partial F(p_0, z_0, x_0)}{\partial p_{n-1}} \\
0 & 0 & \ldots & 0 & \frac{\partial F(p_0, z_0, x_0)}{\partial p_n}
\end{pmatrix}.$$

Since $\frac{\partial F(p_0, z_0, x_0)}{\partial p_n} \neq 0$ this matrix is invertible. The inverse function theorem implies that on the (possibly diminished) neighbourhood Ω of x_0 and suitable $\epsilon > 0$ this map is a twice differentiable homeomorphism $(\Omega \cap H) \times (-\epsilon, \epsilon) \to \Omega$ with twice differentiable inverse mapping. Because we know that the inverse mapping exists, the function $u : \Omega \to \mathbb{R}$ defined in implicit form by

$$u(x(y, s)) = z(y, s) \text{ for all } (y, s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$$

is well-defined.

This function u satisfies the initial conditions of the PDE: we have $x(y, 0) = y$ and so

$$u(y) = u(x(y, 0)) = z(y, 0) = g(y)$$

for all $y \in \Omega \cap H$. It remains to show that u solves the PDE $F(\nabla u(x), u(x), x) = 0$. Observe that the ODEs imply

$$\frac{d}{ds} F(p(y, s), z(y, s), x(y, s)) = 0.$$

Since $F(q(y), g(y), y)$ vanishes for all $y \in \Omega \cap H$ we conclude

$$F(p(y, s), z(y, s), x(y, s)) = 0 \text{ for all } (y, s) \in (\Omega \cap H) \times (-\epsilon, \epsilon).$$

Hence to show that u solves the PDE it suffices to show $p(y, s) = \nabla u(x(y, s))$ for all $(y, s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$.

To this end, we need to establish the following equalities

$$\frac{\partial z(y, s)}{\partial s} = \sum_{j=1}^{n} p_j(y, s) \frac{\partial x_j(y, s)}{\partial s} \qquad \text{and} \qquad \frac{\partial z(y, s)}{\partial y_i} = \sum_{j=1}^{n} p_j(y, s) \frac{\partial x_j(y, s)}{\partial y_i}$$

for all $(y, s) \in (\Omega \cap H) \times (-\epsilon, \epsilon)$ and all $i = 1, \ldots, n - 1$. The first equation follows from the ODE for $x(y, s)$ and $z(y, s)$. For $s = 0$ the second equation follows from the initial conditions for $z(y, s)$, $p(y, s)$ and $x(y, s)$. For $s \neq 0$, let us use $v(y, s)$ for the difference between the left and right hand sides of the second equation:

$$v(y, s) := \frac{\partial z(y, s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y, s) \frac{\partial x_j(y, s)}{\partial y_i}.$$

We need to show that v is always zero. The derivative of the first equation with respect to y_i yields

$$\frac{\partial^2 z(y, s)}{\partial y_i \partial s} = \sum_{j=1}^{n} \left(\frac{\partial p_j(y, s)}{\partial y_i} \frac{\partial x_j(y, s)}{\partial s} + p_j(y, s) \frac{\partial^2 x_j(y, s)}{\partial y_i \partial s} \right).$$

By the commutativity of the second partial derivatives we obtain

$$\frac{\partial}{\partial s}v(y,s) = \frac{\partial^2 z(y,s)}{\partial s \partial y_i} - \sum_{j=1}^{n} \frac{\partial p_j(y,s)}{\partial s}\frac{\partial x_j(y,s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y,s)\frac{\partial^2 x_j(y,s)}{\partial s \partial y_i}$$
$$= \sum_{j=1}^{n}\left(\frac{\partial p_j(y,s)}{\partial y_i}\frac{\partial x_j(y,s)}{\partial s} - \frac{\partial p_j(y,s)}{\partial s}\frac{\partial x_j(y,s)}{\partial y_i}\right)$$
$$= \sum_{j=1}^{n}\frac{\partial p_j(y,s)}{\partial y_i}\frac{\partial F(p(y,s),z(y,s),x(y,s))}{\partial p_j}$$
$$+ \sum_{j=1}^{n}\left(\frac{\partial F(p(y,s),z(y,s),x(y,s))}{\partial x_j} + \frac{\partial F(p(y,s),z(y,s),x(y,s))p_j(y,s)}{\partial z}\right)\frac{\partial x_j(y,s)}{\partial y_i}$$
$$= \frac{\partial}{\partial y_i}F(p(y,s),z(y,s),x(y,s))$$
$$- \frac{\partial F(p(y,s),z(y,s),x(y,s))}{\partial z}\left(\frac{\partial z(y,s)}{\partial y_i} - \sum_{j=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}\right).$$

Notice that the bracketed expression is exactly v. Inserting $F(p(y,s),z(y,s),x(y,s)) = 0$ we obtain

$$\frac{\partial}{\partial s}v(y,s) = -\frac{\partial F(p(y,s),z(y,s),x(y,s))}{\partial z}v(y,s).$$

For each y this is a linear homogeneous ODE for $v(y,s)$ in the variable s with initial value 0 at $s=0$. The unique solution is $v(y,s)\equiv 0$. This implies the second equation for all y and s:

$$\frac{\partial z(y,s)}{\partial y_i} = \sum_{j=1}^{n} p_j(y,s)\frac{\partial x_j(y,s)}{\partial y_i}.$$

Now that we have established the two equalities, we demonstrate that they are not only necessary but also sufficient for the conclusion $p(y,s) = \nabla u(x(y,s))$ for all $(y,s) \in (\Omega \cap H)\times(-\epsilon,\epsilon)$. The solution u is defined as the composition of the inverse of $(y,s)\mapsto x(y,s)$ with $(y,s)\mapsto z(y,s)$. The chain rule implies

$$\frac{\partial u}{\partial x_j} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\frac{\partial z}{\partial y_i}\frac{\partial y_i}{\partial x_j} = \left(\sum_{k=1}^{n} p_k\frac{\partial x_k}{\partial s}\right)\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\left(\sum_{k=1}^{n} p_k\frac{\partial x_k}{\partial y_i}\right)\frac{\partial y_i}{\partial x_j}$$
$$= \sum_{k=1}^{n} p_k\left(\frac{\partial x_k}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\frac{\partial x_k}{\partial y_i}\frac{\partial y_i}{\partial x_j}\right) = \sum_{k=1}^{n} p_k\frac{\partial x_k}{\partial x_j} = p_j.$$

Thus we have shown that the function u, which was constructed from the method of characteristics, solves the PDE.

Theorem 1.8 and the theorem of Picard-Lindelöf imply the uniqueness of the solutions. □

The relation between the method of characteristics as explained in this section and the ad hoc versions we used in previous sections will be explored in the exercises. The important point is they are really the same method, but in many cases the system decouples and the ODEs for x' and z' do not depend on p. This is a nice simplification because it makes solving the p' equations redundant.

1.6 Weak Solutions

In the first few sections situations where there were no solutions, or the method of characteristics gave a 'solution' that was not differentiable. In this section we take a scalar conservation law and look for more general notions of solutions which allow us to extend solutions across the crossing characteristics by allowing a limited amount of non-differentiability. But if we don't have differentiability, what does it meant to satisfy a PDE? For this purpose we use the conserved integrals. Since we will restrict ourselves to the one-dimensional situation for the moment, the natural domains are intervals $\Omega = [a, b]$ with $a < b \in \mathbb{R}$. In this case the conservation law implies

$$\frac{d}{dt}\int_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)).$$

Now we look for functions u with discontinuities along the graph $\{(x,t) \mid x = y(t)\}$ of a C^1-function y. In the case that $y(t)$ belongs to $[a, b]$, we split the integral over $[a, b]$ into the integrals over $[a, b] = [a, y(t)] \cup [y(t), b]$. In such a case let us calculate the derivative of the integral over $[a, b]$:

$$\begin{aligned}\frac{d}{dt}\int_a^b (u(x,t)dx &= \frac{d}{dt}\int_a^{y(t)} u(x,t)dx + \frac{d}{dt}\int_{y(t)}^b u(x,t)dx = \\ &= \dot{y}(t)\lim_{x\uparrow y(t)} u(x,t) + \int_a^{y(t)} \dot{u}(x,t)dx - \dot{y}(t)\lim_{x\downarrow y(t)} u(x,t) + \int_{y(t)}^b \dot{u}(x,t)dx.\end{aligned}$$

We abbreviate $\lim_{x\uparrow y(t)} u(x,t)$ as $u^l(y(t),t)$ and $\lim_{x\downarrow y(t)} u(x,t)$ as $u^r(y(t),t)$ and assume that on both sides of the graph of y the function u is a classical solution of the conservation law:

$$\begin{aligned}&\frac{d}{dt}\int_a^b u(x,t)dx \\ &= \dot{y}(t)(u^l(y(t),t) - u^r(y(t),t)) - \int_a^{y(t)} \frac{d}{dx}f(u(x,t))dx - \int_{y(t)}^b \frac{d}{dx}f(u(x,t))dx \\ &= \dot{y}(t)(u^l(y(t),t) - u^r(y(t),t)) + f(u(a,t)) - f(u(b,t)) + fu^r(y(t),t) - fu^l(y(t),t).\end{aligned}$$

Hence the integrated version of the conservation law still holds, if the following Rankine-Hugonoit condition is fulfilled:

$$\dot{y}(t) = \frac{f(u^r(y,t)) - f(u^l(y,t)}{u^r(y,t) - u^l(y,t)}.$$

Example 1.10. *We consider Burgers equation $\dot{u}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0$ for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ with the following continuous initial values $u(x,0) = g(x)$ and*

$$g(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 1 - x & \text{for } 0 \leq x < 1 \\ 0 & \text{for } 1 \leq x. \end{cases}$$

The first crossing of characteristics happens for $t = 1$:

$$x + tg(x) = \begin{cases} x + t & \text{for } x \leq 0, \\ x + t(1 - x) & \text{for } 0 < x < 1, \\ x & \text{for } 1 \leq x. \end{cases}$$

For $t < 1$ the evaluation at t is a homeomorphism from $\mathbb{R}$ onto itself with inverse

$$x \mapsto \begin{cases} x - t & \text{for } x \leq t, \\ \frac{x-t}{1-t} & \text{for } t < x < 1, \\ x & \text{for } 1 \leq x. \end{cases}$$

Therefore the solution is for $0 < t < 1$ equal to

$$u(x,t) = \begin{cases} 1 & \text{for } x < t, \\ \frac{x-1}{t-1} & \text{for } t < x < 1, \\ 0 & \text{for } 1 \leq x. \end{cases}$$

At $t = 1$ the solutions of the characteristic equations starting at $x \in [0,1]$ all meet at $x = 1$. For $t > 1$ there exists a unique solution satisfying the Rankine-Hugonoit condition, which is 1 on some interval $(\infty, y(t))$ and 0 on the interval $(y(t), \infty)$. The corresponding regions have to be separated by a path with velocity $\frac{1}{2}$ which starts at $(x,t) = (1,1)$. This gives $y(t) = 1 + \frac{t-1}{2}$. For $t \geq 1$ this solution is equal to

$$u(x,t) = \begin{cases} 1 & \text{for } x < 1 + \frac{t-1}{2}, \\ 0 & \text{for } 1 + \frac{t-1}{2} < x. \end{cases}$$

The second initial value problem is not continuous but monotonic increasing. For continuous monotonic increasing functions g the evaluation at t of the solutions of the characteristic equation would be a homeomorphism for all $t > 0$. Therefore in such cases there exists a unique continuous solution for all $t > 0$. But for non-continuous initial values this is not the case.

Example 1.11. *We again consider Burgers equation* $\dot{u}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0$ *for* $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ *with the following non-continuous initial values* $u(x,0) = g(x)$ *and*

$$g(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } 0 < x. \end{cases}$$

Again there is a unique discontinuous solution which is 0 *on some interval* $(-\infty, y(t))$ *and* 1 *on the interval* $(y(t), \infty)$. *By the Rankine-Hugonoit condition both regions are separated by a path with velocity* $\frac{1}{2}$. *This solution is equal to*

$$u(x,t) = \begin{cases} 0 & \text{for } x < \frac{t}{2}, \\ 1 & \text{for } \frac{1}{2} < x. \end{cases}$$

But there exists another continuous solution, which clearly also satisfies the Rankine-Hugonoit condition:

$$u(x,t) = \begin{cases} 0 & \text{for } x \leq 0, \\ \frac{x}{t} & \text{for } 0 < x < t, \\ 1 & \text{for } t \leq x. \end{cases}$$

These solutions are constant along the lines $x = ct$ *for* $c \in [0,1]$. *These lines all intersect in the discontinuity at* $(x,t) = (0,0)$. *Besides these two extreme cases there exists infinitely many other solutions with several regions of discontinuity, which all satisfy the Rankine-Hugonoit condition.*

These examples show that such weak solutions exists for all $t \geq 0$ but are not unique. We now restrict the space of weak solutions such that they have a unique solutions for all $t \geq 0$. Since we want to maximise the regularity we only accept discontinuities if there are no continuous solutions. In the last example we prefer the continuous solution. So for Burgers equation this means we only accept discontinuous solutions, which take larger values for smaller x and smaller values for larger x.

Definition 1.12 (Lax Entropy condition)**.** *A discontinuity of a weak solution along a* C^1*-path* $t \mapsto y(t)$ *satisfies the Lax entropy condition, if along the path the following inequality is fulfilled:*

$$f'(u^l(y,t)) > \dot{y}(t) > f'(u^r(y,t)).$$

A weak solutions with discontinuities along C^1*-paths is called an admissible solution, if along the path both the Rankine-Hugonoit condition and the Lax Entropy condition are satisfied.*

For continuous g there is a crossing of characteristics if $f'(g(x_1)) > f'(g(x_2))$ for $x_1 < x_2$. So this condition ensures that discontinuities can only show up if we cannot avoid a crossing of characteristics.

Theorem 1.13. *Let $f \in C^1(\mathbb{R},\mathbb{R})$ be convex and u and v two admissible solutions of*

$$\dot{u}(x,t) + f'(u(x,t))\frac{\partial u}{\partial x}(x,t) = 0.$$

in $L^1(\mathbb{R})$. Then $t \mapsto \|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})}$ is monotonically decreasing.

Proof. We divide $\mathbb{R}$ into maximal intervals $I = [a(t), b(t)]$ with the property that either $u(x,t) > v(x,t)$ or $v(x,t) > u(x,t)$ for all $x \in (a(t), b(t))$. This means that either $x \mapsto u - v$ vanishes at the boundary, or is discontinuous and changes sign at the boundary. We claim that the boundaries $a(t)$ and $b(t)$ of these maximal intervals are differentiable. We prove this only for $a(t)$. For $b(t)$ the proof is analogous. To simplify notation we write a and b instead of $a(t)$ and $b(t)$. If either $u(\cdot,t)$ or $v(\cdot,t)$ is discontinuous at a, then by definition of an admissible solution the locus of the discontinuity a is differentiable with respect to t. If u and v are both continuously differentiable at (a,t) with $u(a,t) = v(a,t)$, then by the method of characteristic for sufficiently small $\epsilon > 0$ all $x \in (a-\epsilon, a+\epsilon)$ with $u = v$ preserve this property along characteristic lines $x + tf'(u(x(t),t) = x + tf'(v((x(t),t)$. So along these lines also the properties $u \neq v$ and $u > v$ are preserved. This implies that a is differentiable with $\dot{a}(t) = f'(u(a,t)) = f'(v(a,t))$. We only consider intervals on whose interior $u > v$. On the other intervals these arguments apply with interchanged u and v. Now we calculate

$$\begin{aligned}
&\frac{d}{dt}\int_{a(t)}^{b(t)} (u(x,t) - v(x,t))dx \\
&\qquad = \int_{a(t)}^{b(t)} (\dot{u}(x,t) - \dot{v}(x,t))dx + \dot{b}(u(b,t) - v(b,t)) - \dot{a}(t)(u((a,t) - v(a,t)) \\
&\quad = \int_{a(t)}^{b(t)} \frac{d}{dx}(f(v(x,t) - f(u(x,t))dx + \dot{b}(t)(u(b,t) - v(b,t)) - \dot{a}(t)(u((a,t) - v(a,t)) \\
&= f(v(b,t) - f(u(b,t) + \dot{b}(t)(u(b,t) - v(b,t)) + f(u(a,t) - f(v(a,t) + \dot{a}(t)(v(a,t) - u(a,t)).
\end{aligned}$$

If u and v are both differentiable at (a,t), then they take the same values at (a,t) and the corresponding terms in the last line vanishes. The same holds, if u and v are both differentiable at (b,t). For convex f the derivative f' is monotonically increasing and the Lax-Entropy condition implies at all discontinuities y of $u(\cdot,t)$ and $v(\cdot,t)$

$$u^l(y,t) > u^r(y,t), \qquad\qquad v^l(y,t) > v^r(y,t),$$

respectively. If one of the two solutions u and v is at the boundary of I continuous and the other is non-continuous, then the value of the continuous solution belongs to the closed interval between the limits of the non-continuous solution, because at the boundary either $u - v$ becomes zero or changes sign. For v being continuous and u being discontinuous at a we would have $u^l(a,t) \leq v(a,t) \leq u^r(a,t)$ by $u > v$ on (a,b) in contradiction to the former inequality. So either $u(\cdot,t)$ is continuous and differentiable at a and $v(\cdot,t)$ is discontinuous at $a(t)$ and analogously u is discontinuous at b and v is continuous and differentiable at b. The Rankine Hugonoit condition determines $\dot{a}(t)$ and $\dot{b}(t)$. At $a(t)$ the corresponding contribution to $\frac{d}{dt}\|u(\cdot,t)-v(\cdot,t)\|_1$ is

$$\begin{aligned} f(u(a,t)) - f(v^r(a,t)) + \dot{a}(t)\,(v^r(a,t) - u(a,t)) = \\ = f(u(a,t)) - f(v^r(a,t)) + \frac{f(v^r(a,t)) - f(v^l(a,t))}{v^r(a,t) - v^l(a,t)}\,(v^r(a,t) - u(a,t)) \\ = f(u(a,t)) - \left(f(v^r(a,t))\frac{v^l(a,t) - u(a,t)}{v^l(a,t) - v^r(a,t)} + f(v^l(a,t))\frac{u(a,t) - v^r(a,t)}{v^l(a,t) - v^r(a,t)} \right). \end{aligned}$$

Since f is convex the secant lies above the graph of f. Since $u(a,t) \in [v^r(a,t), v^l(a,t)]$ this expression is non-positive. At $b(t)$ this contribution is

$$\begin{aligned} f(v(b,t)) - f(u^l(b,t)) + \dot{b}(t)\,\big(u^l(b,t) - v(b,t)\big) = \\ = f(v(b,t)) - f(u^l(b,t)) + \frac{f(u^r(b,t)) - f(u^l(b,t))}{u^r(b,t) - u^l(b,t)}\,\big(u^l(b,t) - v(b,t)\big) \\ = f(v(b,t)) - \left(f(u^r(b,t))\frac{u^l(b,t) - v(b,t)}{u^l(b,t) - u^r(b,t)} + f(u^l(b,t))\frac{v(b,t) - u^r(b,t)}{u^l(b,t) - u^r(b,t)} \right). \end{aligned}$$

Again due to $v(b,t) \in [u^r(b,t), u^l(b,t)]$ this expression is non-positive.

If finally both solutions are discontinuous at $a(t)$ or $b(t)$. Since $u(\cdot,t) - v(\cdot,t)$ is positive on I, the Lax Entropy condition implies $[u^r(a,t), u^l,(a,t)] \subset [v^r(a,t), v^l(a,t)]$ and $[v^r(b,t), v^l(b,t)] \subset [u^r(b,t), u^l(b,t)]$, respectively. The corresponding contributions to $\frac{d}{dt}\|u(\cdot,t)-v(\cdot,t)\|_1$ are again non-positive:

$$\begin{aligned} f(u^r(a,t)) - f(v^r(a,t)) + \dot{a}(t)\,(v^r(a,t) - u^r(a,t)) = \\ = f(u^r(a,t)) - f(v^r(a,t)) + \frac{f(v^r(a,t)) - f(v^l(a,t))}{v^r(a,t) - v^l(a,t)}\,(v^r(a,t) - u^r(a,t)) \\ = f(u^r(a,t)) - \left(f(v^r(a,t))\frac{v^l(a,t) - u^r(a,t)}{v^l(a,t) - v^r(a,t)} + f(v^l(a,t))\frac{u^r(a,t) - v^r(a,t)}{v^l(a,t) - v^r(a,t)} \right). \end{aligned}$$

$$\begin{aligned} f(v^l(b,t)) - f(u^l(b,t)) + \dot{b}(t)\,\big(u^l(b,t) - v^l(b,t)\big) = \\ = f(v^l(b,t)) - f(u^l(b,t)) + \frac{f(u^r(b,t)) - f(u^l(b,t))}{u^r(b,t) - u^l(b,t)}\,\big(u^l(b,t) - v^l(b,t)\big) \\ = f(v^l(b,t)) - \left(f(u^r(b,t))\frac{u^l(b,t) - v^l(b,t)}{u^l(b,t) - u^r(b,t)} + f(u^l(b,t))\frac{v^l(b,t) - u^r(b,t)}{u^l(b,t) - u^r(b,t)} \right). \end{aligned}$$

Hence the contributions to $\frac{d}{dt}\|u(\cdot,t)-v(\cdot,t)\|_1$ of all intervals are non-positive. □

This implies that admissible solutions are unique, if they exist. By utilising an explicit formula for admissible solutions one can also prove the existence of admissible solutions.

The following theorem is Theorem 10.3 in the lecture notes "Hyperbolic Partial Differential Equations" by Peter Lax, Courant Lecture Notes in Mathematics **14**, American Mathematical Society (2006), which also supplies a proof.

Theorem 1.14. *For $f \in C^2(\mathbb{R}, \mathbb{R})$ is strictly convex and $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ there exists an unique admissible solution $u(x,t)$ of*

$$\dot{u}(x,t) + f'(u(x,t))\frac{\partial u}{\partial x}(x,t) = 0 \quad \textit{and} \quad u(x,0) = g(x) \quad \textit{for all } x \in \mathbb{R}.$$

Chapter 2

General Concepts

2.1 Classification of Second order PDEs

For PDEs of order greater than one, there does not exists a general theory. We shall present in Section 2.2 an example of a PDE with smooth coefficients, which has in a neighbourhood of some point no solutions at all. Over the time there have been discovered different methods to solve several PDEs, in particular those PDEs which show up in physics. Afterwards these methods were extended to larger and larger classes of PDEs. It turned out that the successful methods of solving PDEs differ from each other substantially. As a result there does not exists one unified theory of PDEs, but there exist several islands of well understood families of PDEs inside the large set of all PDEs. It was Jacobi who formulated in his lectures on Dynamics in the years 1842-43 the following general recipe:

“The main obstacle for the integration of a given differential equations lies in the definition of adapted variables, for which there is no general rule. For this reason we should reverse the direction of our investigation and should endeavour to find, for a successful substitution, other problems which might be solved by the same.”

The strategy is to determine for any successful method all PDEs which can be solved by this method. We have seen that the method of characteristics is a more-or-less general method to solve first order PDEs. Now we investigate the second order PDEs. In this lecture we consider only second order linear PDEs. A general second order linear PDE has the following form

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u + \sum_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x) = 0.$$

By Schwarz’s Theorem for twice differentiable u this expression does not change if we replace a_{ij} by $\frac{1}{2}(a_{ij} + a_{ji})$. So we may assume that a_{ij} is symmetric and diagonalizable.

Elliptic PDEs. If the matrix a_{ij} is the unity matrix and $b = 0 = c$, then this is the

Laplace equation. $$\triangle u := \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

Solutions of the Laplace equation are called harmonic functions. In Chapter 3 we present several tools which establish many properties of these harmonic functions. It turns out that many properties of the harmonic functions also apply to general solutions of $Lu = 0$, if the matrix a_{ij} is positive (or negative) definite. These are the main examples of the so called elliptic PDEs. There has been done a lot of work to extend these tools to larger and larger classes of elliptic PDEs. One of the results is that the influence of the higher order derivatives on the properties of solutions is much more important than the influence of the lower order derivatives. An important tool are so called a priori estimates. Such estimates show that the lower order derivatives can be estimated in terms of the second order derivatives. We offer another lecture which presents many of these tools for such elliptic second order PDEs.

Beside the linear elliptic PDEs there are also non-linear PDEs, to which these methods of elliptic PDEs apply. An important example whose investigation played a major role in the development of the elliptic theory is the

Minimal surface equation. $$\nabla \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0, \qquad u : \Omega \to \mathbb{R}, \qquad \Omega \subset \mathbb{R}^n \text{ open}.$$

The graphs of solutions describe so called minimal surfaces. The area of such hypersurfaces in $\mathbb{R}^{n+1}$ does not change with respect to infinitesimal variations. Soap bubbles are examples of such minimal surfaces. The boundary value problem of the minimal surface equation is called Plateau's problem. For the first proof of the existence of solutions of this Plateau problem in the 1930s, Jesse Douglas received the first Field's Medal. In this non-linear second order PDE the coefficients of the second derivatives also depend on the solution. A lot of work has been done to extend the tools of elliptic theory to elliptic PDEs whose coefficients belong to larger and larger functions spaces. This development induced the introduction of many new function spaces. In Section 2.6 we shall introduce the so called space of distributions. Many of the more advanced functions spaces are build on the base of these spaces.

Parabolic PDEs. For these linear PDEs the matrix a_{ij} considered as a symmetric bilinear form is only semi-definite and they belong to the boundary of the class of elliptic PDEs. Most of the methods of elliptic PDEs have an extension to this limiting case. So these limiting cases together with the class of elliptic PDEs form some extended class of elliptic PDEs. Of particular importance is the subclass of linear PDEs with semi-definite matrices a_{ij} which have a one-dimensional kernel. Since symmetric matrices are always diagonalizable this means that one eigenvalue of a_{ij} vanishes and all other eigenvalues have the same sign. In spite of the deep relationship to the elliptic PDEs these equations have their own label: parabolic PDEs. The simplest example is the

Heat equation. $\dot{u} - \triangle u = 0.$

These parabolic PDEs describe diffusion processes. These are processes which level inhomogeneities of some quantity by some flow along the negative gradient of the quantity. A typical example for this quantity is the temperature from which the name for the heat equation originates. Many stochastic processes have this property. So the theory of parabolic PDEs has a deep relationship to the theory of stochastic processes. In this lecture we present in Chapter 4 this simplest example of linear parabolic PDE. We shall see how the tools for the Laplace equation can be applied in modified form to this heat equation. In case of the parabolic PDEs there too exists a non-linear example from the geometric analysis, whose investigation played a major role for the development of the elliptic theory (the tensor fields g and R are defined below):

Ricci Flow. $\dot{g}_{ij} = -2R_{ij}.$

This PDE describes a diffusion-like process on Riemannian manifolds. It levels the inhomogeneities of the metric, namely the Riemannian metric g. In the long run the corresponding Riemannian manifolds converge to metric spaces with large symmetry groups. Richard Hamilton proposed (in the 1970s) a program that aims to prove the geometrization conjecture of Thurston with the help of these PDEs. It states that every three-dimensional manifold can be split into parts, which can be endowed with an Riemannian metric such that the isometry group acts transitively. This conjecture implies the Poincare conjecture, which states that every simply connected compact manifold is the 3-sphere. Hamilton tries to control the long time limit of the Ricci flow on a general 3-dimensional Riemannian manifold. In 2003 the Russian mathematician Grisha Perelman published on the internet three articles which overcome the last obstacle of this program. This lead to the first proof of one of the Millennium Problems of the American Mathematical Society and was a great success of geometric analysis.

Hyperbolic PDEs. Besides the elliptic PDEs (including the limiting cases) the second important class of linear PDEs are called hyperbolic. In this case the matrix a_{ij} has one eigenvalue of opposite sign than all other eigenvalues. The simplest example is the

Wave equation. $\frac{\partial^2 u}{\partial t^2} - \triangle u = 0.$

In Chapter 5 we present the methods how to solve this equation. We shall see that it describes the propagation of waves with constant finite speed. The solutions of general hyperbolic equations are similar to the solutions of this case, and many tools can be generalised to all hyperbolic PDEs. The investigation of these PDEs depend on the understanding of all trajectories, which propagate by the given speed. It was motivated by the theory of the electrodynamic fields, whose main system of PDEs are the

Maxwell equations.
$$\begin{aligned} \dot{E} - \nabla \times B &= -4\pi j & \dot{B} + \nabla \times E &= 0 \\ \nabla \cdot E &= 4\pi\rho & \nabla \cdot B &= 0. \end{aligned}$$

In this theory there is given a distribution of charges ρ and currents j on space time

$\mathbb{R} \times \mathbb{R}^3$. The unknown functions are the electric magnetic fields E and B, which describe the electrodynamic forces induced by the given distributions of charges and currents ρ and j. The conservation of charge is formulated in the same way as in the scalar conservation law. So the change of the total charge contained in a spatial domain is described by the flux of the current through the boundary of the domain. By the divergence theorem this means that distributions of charge ρ and currents j obey

$$\dot{\rho} + \nabla \cdot j = 0.$$

Again there exists a non-linear version which stimulated the development of the theory:

Einsteins field equations of general relativity. $R_{ij} - \frac{1}{2} g_{ij} R = \kappa T_{ij}.$

Here for a given distribution of masses the energy stress tensor and the space time metric g_{ij} are the unknown functions. This metric is a symmetric bilinear form with one positive and three negative eigenvalues on the tangent space of space time. The corresponding Ricci curvature is denoted by R_{ij} and the scalar curvature by R:

$$\Gamma^k_{ij} := \frac{1}{2} \sum_{l=0}^{3} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right), \quad \left(g^{ij}\right) := \left(g_{ij}\right)^{-1} \text{ inverse metric}$$

$$R_{ij} := \sum_{k=0}^{3} g^{kl} \left(\frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{ik}}{\partial x^j} + \sum_{l=0}^{3} \left(\Gamma^k_{lk} \Gamma^l_{ij} - \Gamma^k_{lj} \Gamma^l_{ik} \right) \right), \quad R := \sum_{i,j=0}^{3} g^{ij} R_{ij}.$$

Integrable Systems with Lax operators. Finally I want to mention a smaller class of PDEs, which are the main objects of my research. They are non-linear PDEs which describe an evolution with respect to time which is very stable. This means that the solutions have in a specific sense a maximal number of conserved quantities. The theory of integrable systems belongs to the field of Hamiltonian mechanics, which originated from Newtons description of the motion of the planets. The Scottish Lord John Scott Russell got very excited in 1934 about the observation of an solitary wave in a Scottish channel and published a "Report on Waves". This report was quite influential. The two Dutch mathematicians Korteweg and De Vries translated his observation into a PDE describing the profile of water waves travelling along the channel:

Korteweg-de-Vries equation. $4\dot{u} - 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = 0.$

First by numerical experiments in the 1950s with the first computers and latter in the 1970s by mathematical theory, the solutions of this PDE were shown to have exactly the properties which made Lord Russell so exited: they describe waves which propagate through each other without changing their shape. This lead to the discovery of an hidden relation of the theory of integrable systems with the theory of Riemann surfaces, which is another field with a long history. A major step towards the discovery of this relation was the observation of Peter Lax that this equation can be written as

$$\dot{L} = [A, L] \qquad \text{with} \qquad L := \frac{\partial^2}{\partial x^2} + u \qquad A := \frac{\partial^3}{\partial x^3} + \frac{3u}{2}\frac{\partial}{\partial x} + \frac{3}{4}\frac{\partial u}{\partial x}.$$

2.2 Existence of Solutions

In order to demonstrate the difference between ODEs and PDEs we shall present an example of a partial differential equation with smooth coefficients without solutions. This example is a simplification by Nirenberg of an example of H. Lewy.

For a given complex-valued function f on a domain $(x, y) \in \mathbb{R}^2$ we look for a complex valued solution u on the same domain of the following differential equations:

$$\frac{\partial u}{\partial x} + \imath x \frac{\partial u}{\partial y} = f(x, y).$$

We impose the following two conditions on the smooth function f:

(i) $f(-x, y) = f(x, y)$

(ii) there exists a sequence of positive numbers $\varrho_n \downarrow 0$ converging to zero, such that f vanishes on a neighbourhood of the circles $\partial B(0, \varrho_n)$ in contrast to non-vanishing integrals $\int_{B(0,\varrho_n)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y \neq 0$.

If $h : \mathbb{R} \to [0, \infty)$ is a smooth periodic function vanishing on an interval but not on $\mathbb{R}$, then $f(x) := \exp(-1/|x|)h(1/|x|)$ for $x \neq 0$ and $f(0) = 0$ has these two properties.

Now we shall prove by contradiction that there exists no continuously differentiable solution u in a neighbourhood of $(0, 0) \in \mathbb{R}^2$.

Step 1: If the function $u(x, y)$ is a solution, then due to (i) $-u(-x, y)$ is also a solution. Hence we may replace $u(x, y)$ by $\frac{1}{2}(u(x, y) - u(-x, y))$ and assume $u(-x, y) = -u(x, y)$.

Step 2: We claim that every solution u vanishes on the circles $\partial B(0, \varrho_n)$. In fact, we transform small annuli A onto domains $\tilde{A}$ in $\mathbb{R}^2$:

$$A \to \tilde{A}, \qquad (x, y) \mapsto \begin{cases} (x^2/2, y) & \text{for } x \geq 0 \\ (-x^2/2, y) & \text{for } x < 0. \end{cases}$$

These transformations are homeomorphisms from A onto $\tilde{A}$. On the subdomains $\tilde{A}_+ = \left\{(s, y) \in \tilde{A} \mid s > 0\right\}$ the function $\tilde{u}(s, y) = u(x^2/2, y)$ is holomorphic:

$$2\bar{\partial}\tilde{u} = \frac{\partial \tilde{u}(s, y)}{\partial s} + \imath \frac{\tilde{u}(s, y)}{\partial y} = \frac{dx}{ds}\frac{\partial u(x, y)}{\partial x} + \imath \frac{\partial u(x, y)}{\partial y} = \frac{1}{x}\left(\frac{\partial u(x, y)}{\partial x} + \imath x \frac{\partial u(x, y)}{\partial y}\right) = 0.$$

Due to step 1. the function $\tilde{u}$ vanishes on the line $s = 0$. This implies that $\tilde{u}$ together with the Taylor series vanishes identically on $\tilde{A}_+$ and due to step 1 on $\tilde{A}$.

Step 3: The Divergence Theorem yields a contradiction to the assumption (ii):

$$\begin{aligned}\int_{B(0,\varrho_n)} f \,\mathrm{d}x\,\mathrm{d}y &= \int_{B(0,\varrho_n)} \left(\frac{\partial u}{\partial x} + \imath x \frac{\partial u}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int_{B(0,\varrho_n)} \nabla \cdot \begin{pmatrix} u \\ \imath x u \end{pmatrix} \mathrm{d}x\,\mathrm{d}y \\ &= \int_{\partial B(0,\varrho_n)} \begin{pmatrix} u \\ \imath x u \end{pmatrix} \cdot N(x,y)\,\mathrm{d}\sigma(x,y) = 0,\end{aligned}$$

Therefore the given differential equation has no continuously differentiable solution.

This example also implies that the following partial differential equation with smooth real coefficients has no four times differentiable real solution:

$$\left(\frac{\partial}{\partial x} + \imath x \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \imath x \frac{\partial}{\partial y}\right)^2\left(\frac{\partial}{\partial x} + \imath x \frac{\partial}{\partial y}\right)\tilde{u} = \left(\left(\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}\right)^2 + \frac{\partial^2}{\partial y^2}\right)\tilde{u} = f.$$

Here f is a real smooth function with the properties (i) and (ii). For any real solution $\tilde{u}$, the following complex function would be a solution of the complex PDE:

$$u = \left(\frac{\partial}{\partial x} - \imath x \frac{\partial}{\partial y}\right)^2\left(\frac{\partial}{\partial x} + \imath x \frac{\partial}{\partial y}\right)\tilde{u}.$$

2.3 Regularity of Solutions

The regularity of a solution of a differential equation refers to the local properties of the corresponding functions. The most general functions we shall consider are distributions, which we say have the lowest regularity. They contain the measurable functions with the next highest regularity. The elements of L^p_{loc} describe ever smaller families of functions, whose regularity increase with $p \in [1,\infty]$. The next smallest class are Sobolev functions whose k-th order partial derivatives belong to L^p_{loc}. The regularity further increases for the functions in C^k. Finally we end with the smooth functions and the analytic functions with the highest regularity.

2.4 Boundary Value Problems

Our investigations of solutions of partial differential equations aims for a complete characterisations of all solutions. In general partial differential equations have an infinite dimensional space of solutions. A solution of an ordinary differential equations of m-th order is in many cases uniquely determined by fixing the values of the first m derivatives at some initial value of the parameter. For partial differential equations we search a similar characterisation. The solutions are functions on higher dimensional domains $\Omega \subset \mathbb{R}^n$. A natural condition is the specification of the values of the solution and some of its derivatives on the boundary of the domain. The search for solutions which obey this further specification are called boundary value problems. So an important objective in the investigation of partial differential equations is to find boundary value problems that have unique solutions. If we determine in addition all possible boundary values that have solutions, then the space of solutions is completely parameterised.

2.5 Divergence Theorem

In this section we present a generalisation of the fundamental theorem of calculus to higher dimensions, namely the divergence theorem. This theorem has many important consequences. In this section we present two: First we generalise partial integration to higher dimensions. Second we explain in which sense the higher dimensional scalar conservation law describes a conserved quantity.

The divergence theorem is a statement about the integral over a submanifold of $\mathbb{R}^n$, so naturally we should define submanifolds and their integrals. We begin by defining the integral on a regularly parameterised subset.

Definition 2.1. *A continuously differentiable homeomorphisms $\Phi : U \subset \mathbb{R}^k \to A \subset \mathbb{R}^n$ is called a (k-dimensional) parameterisation of A. It is called regular if the Jacobian Φ' has rank k at every point of U.*

The Jacobian of Φ is an $n \times k$ matrix, whose rank cannot be greater than n, so $1 \leq k \leq n$. The idea is that we view A as a piece of the lower dimensional space $\mathbb{R}^k$ embedded into $\mathbb{R}^n$. For an example of a non-regular parameterisation, consider the parameterisation $(x, y) \mapsto (x, 0, 0)$ of the x-axis in $\mathbb{R}^3$. We see that y is not really playing any role and the parameterised set is only one-dimensional, not two-dimensional as we would expect. This is the reason we should consider regular parameterisations.

Definition 2.2. *Let $A \subset \mathbb{R}^n$ be a subset with a regular parameterisation Φ and f a continuous function on A. We define*

$$\int_A f \; \mathrm{d}\sigma := \int_U f \circ \Phi \sqrt{\det((\Phi')^T \Phi')} \, \mathrm{d}\mu_{\mathbb{R}^k} \, .$$

The symbol $\mathrm{d}\sigma$ can be given a formal meaning, but for us it is just a reminder that it is a 'surface integral' and not an integral on a subset of $\mathbb{R}^n$ in the usual sense. The k-dimensional parallelotope spanned by the k column vectors of a $n \times k$-matrix A has the volume $\sqrt{\det(A^T A)}$. The motivation for the $\sqrt{\det}$ factor in the definition of the integral is that it measures the distortion of the parameterisation. This value turns out to be independent of the choice of regular parameterisation of A. Suppose that we have another regular parameterisation Ψ of A. Then we have a homeomorphism $\Upsilon = \Phi^{-1} \circ \Psi : \Psi^{-1}[A] \to \Phi^{-1}[A]$. We claim that Υ is continuously differentiable. This is not so clear, because Φ^{-1} is only defined on A and we can't apply the chain rule directly.

The idea is roughly speaking to approximate A by a linear space so that we can apply the chain rule. For any $x \in A$ let V be the k-dimensional linear subspace that is $x + \operatorname{img} \Phi'$ and let P be the orthogonal projection of $\mathbb{R}^n$ onto V. With these definitions the derivative of composition $P \circ \Phi : U \to V$ has full rank at $\Phi^{-1}(x)$ and $P \circ \Phi(\Phi^{-1}(x)) = x$. Therefore for a neighbourhood V' of x in V and U' of $\Phi^{-1}(x)$ in U there exists a continuously differentiable inverse function $(P \circ \Phi)^{-1} : V' \to U'$, due to the inverse function theorem. Let $A' = \Phi[U']$. Then

$$(P \circ \Phi)^{-1} \circ (P \circ \Phi) = \mathrm{id}_{U'} \quad \Rightarrow \quad (P \circ \Phi)^{-1} \circ P|_{A'} = \Phi^{-1}|_{A'}.$$

Thus we can write

$$\Upsilon|_{\Psi^{-1}[A']} = \Phi^{-1} \circ \Psi|_{\Psi^{-1}[A']} = (P \circ \Phi)^{-1} \circ P \circ \Psi|_{\Psi^{-1}[A']}$$

as the composition of three continuously differentiable functions. Because we can do this at every point $x \in A$ it follows that Υ is continuously differentiable at every point.

Now we can carry out an computation that connects the two integrals

$$\begin{aligned}
&\int_{\Psi^{-1}[A]} f \circ \Psi \sqrt{\det((\Psi')^T \Psi')} \, \mathrm{d}\mu_{\mathbb{R}^k} \\
&= \int_{\Psi^{-1}[A]} f \circ \Phi \circ \Upsilon \sqrt{\det(((\Phi \circ \Upsilon)')^T (\Phi \circ \Upsilon)'} \, \mathrm{d}\mu_{\mathbb{R}^k} \\
&= \int_{\Psi^{-1}[A]} \left(f \circ \Phi \sqrt{\det((\Phi')^T \Phi')} \right) \circ \Upsilon | \det \Upsilon' | \, \mathrm{d}\mu_{\mathbb{R}^k} \\
&= \int_{\Phi^{-1}[A]} f \circ \Phi \sqrt{\det((\Phi')^T \Phi')} \, \mathrm{d}\mu_{\mathbb{R}^k}.
\end{aligned}$$

In the last step we applied the transformation formula of Jacobi.

This is a very practical definition in that it gives you a concrete integral to compute. However many subsets that we want to consider cannot be regularly parameterised. Usually this is because they cannot be covered by a single parameterisation. The typical example is the sphere: any open set $U \subset \mathbb{R}^k$ is not compact and the sphere is compact, so there cannot exist a homeomorphism Φ between them. However if we use two parameterisations, then each can cover a part of the sphere and together they can cover the whole sphere. This motivates the following definition of submanifold.

Definition 2.3. *A subset $A \subset \mathbb{R}^n$ is called a k-dimensional submanifold if there exists subsets A_i such that each A_i has a regular k-dimensional parameterisation and $A = \cup A_i$. A submanifold is called embedded if for each A_i there is an open subset O_i of $\mathbb{R}^n$ such that $A \cap O_i = A_i$.*

The trouble is now these A_i can overlap, so if we just integrate in each parameterisation then we will 'double-count' the points of A. The answer to this is an elegant theoretical tool, but one that is not practically useful: a so called *partitions of unity.*

Definition 2.4. *(Partition of Unity) Let $\Omega \subset \mathbb{R}^n$ be covered by a countable family $(U_i)_{i\in\mathbb{N}}$ of open subsets of $\mathbb{R}^n$, i.e. $\bigcup_{i\in\mathbb{N}} U_i = \Omega$. A smooth partition of unity is a countable family $(h_i)_{i\in\mathbb{N}}$ of smooth functions $h_i : \Omega \to [0,1]$ with the following properties:*

(i) *Each $x \in \Omega$ has a neighbourhood on which all but finitely h_i vanish identically.*

(ii) *For all $x \in \Omega$ we have $\sum_{i=1}^{\infty} h_i(x) = 1$.*

(iii) *Each h_i vanishes outside of U_i.*

For every family of open subsets of $\mathbb{R}^n$ there exists a smooth partition of unity. A proof can be found in many textbooks and in Prof Schmidt's script of the lecture Analysis II.

Definition 2.5. *Let $A \subset \mathbb{R}^n$ be compact and an embedded k-dimensional submanifold. Let f be a continuous function on A. Because A is compact and $A \subset \bigcup O_i$, only finitely many parameterisations are needed to cover it. Choose a partition of unity $(h_i)_{i\in\mathbb{N}}$ subordinate to O_i. We define*

$$\int_A f \,\mathrm{d}\sigma = \sum_i \int_{A_i} h_i f \,\mathrm{d}\sigma.$$

The idea of this definition is that we can write $f(x) = 1 \times f(x) = \sum_i h_i(x) f(x)$. Then each function $h_i f$ is zero outside of A_i so it is only necessary to integrate it on A_i, not on all of A. We assumed that A was compact so that the sum is finite and we avoid any issues of convergence. The restriction that A is compact is not necessary, but then one must deal with the convergence issues.

Lemma 2.6. *The integral $\int_A f \,\mathrm{d}\sigma$ neither depends on the choice of the partition of unity not on the choice of the parametrizations.*

Proof. Suppose that we have two covers of parameterising sets $A = \cup_i A_i = \cup_j B_j$ and correspondingly two partitions of unity h_i and g_j. Define a new cover $C_{i,j} = A_i \cap B_j$. It has a partition of unity $h_i g_j$. Each set $C_{i,j}$ can be parameterised by restricting the parameterisation Φ of A_i to $\Phi^{-1}[A_i \cap B_j]$. Observe

$$\sum_i \int_{A_i} h_i f \,\mathrm{d}\sigma = \sum_i \int_{A_i} \left(\sum_j g_j \right) h_i f \,\mathrm{d}\sigma = \sum_{i,j} \int_{A_i} g_j h_i f \,\mathrm{d}\sigma = \sum_{i,j} \int_{C_{i,j}} g_j h_i f \,\mathrm{d}\sigma.$$

The same calculation holds for the integral $\sum_j \int_{B_j} g_j f \,\mathrm{d}\sigma$, showing that the two are equal. We have already seen that if we use two different parameterisations for the same set that the integral has the same value. Therefore we have shown that definition is independent of parameterisation and partition of unity. □

In the divergence theorem we consider open subsets $\Omega \subset \mathbb{R}^n$ whose boundary are $(n-1)$-dimensional submanifolds. Below the Definition 2.2 we constructed a projection from A' onto a subset V' the hyperplane V that was a homeomorphism. If you look at the argument, it also works for any linear subspace V such that $P \circ \Phi$ has full rank. In particular in the $(n-1)$ dimensional case it must hold for at least one of the coordinate hyperplanes. Without loss of generality we can assume that $V = \mathbb{R}^{n-1} \times \{x_n\}$. If we view this projection in reverse, we understand that A' is the graph of a function over V', it has the form $v \in V' \subset \mathbb{R}^{n-1} \mapsto (v, \lambda(v))$ for a smooth function $\lambda : V' \to \mathbb{R}$. This allows us to locally write a formula for the normal vector

$$N = \pm \frac{1}{\sqrt{1 + |\nabla\lambda|^2}} \begin{pmatrix} -\nabla\lambda \\ 1 \end{pmatrix}.$$

We see that it a smooth vector field, well-defined up to a choice of sign. Moreover, the graph itself actually defines a parameterisation of A'. With respect to this parameterisation $\Phi(v) = (v, \lambda(v))^T$ we have $\Phi' = (I|\nabla\lambda)^T$. The follow calculation makes use of the Weinstein–Aronszajn identity

$$\begin{aligned} \det(\Phi')^T\Phi' = \det(I|\nabla\lambda)^{TT}(I|\nabla\lambda)^T &= \det(I_{n-1} + \nabla\lambda\nabla\lambda^T) \\ &= \det(I_1 + \nabla\lambda^T\nabla\lambda) = 1 + |\nabla\lambda|^2. \end{aligned}$$

Theorem 2.7. *(Divergence Theorem) Let $\Omega \subseteq \mathbb{R}^n$ be bounded and open with $\partial\Omega$ being a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. Let $F : \bar{\Omega} \to \mathbb{R}^n$ be continuous and differentiable on Ω such that ∇F continuously extends to $\partial\Omega$. Then we have*

$$\int_\Omega \nabla \cdot F \,\mathrm{d}\mu = \int_{\partial\Omega} F \cdot N \,\mathrm{d}\sigma$$

where N is the outward-pointing normal.

Proof. First we consider the case that F and F' are zero on $\partial\Omega$. The right hand side of the divergence theorem is zero. By defining it to be zero outside of Ω we can extend F to a continuously differentiable function on $\mathbb{R}^n$. Choose a Cartesian product of finite intervals which contains Ω. The continued function vanishes on the boundary of this box. By Fubini we may integrate the i-th term of $\nabla \cdot F = \partial_1 F_1 + \ldots, +\partial_n F_n$ first over dx_i. Due to the fundamental theorem of calculus this integral is the difference of the values of F at two boundary points and vanishes. This shows that the left side of the divergence theorem also vanishes.

For the general case we cover $\bar{\Omega}$ by Ω and open subsets $V' \times (a, b) \subset \mathbb{R}^n$ as described above, one for every point of $\partial\Omega$. Due to the compactness of $\bar{\Omega}$ we can find a finite subcover and choose a subordinate partition of unity. This decomposes f into a finite sum $\sum h_l F$. By linearity it suffices to show the statement for any $h_l F$ individually. The $h_l F$ corresponding to Ω has already been dealt with, so we only need to handle the 'boundary terms'.

We relabel the coordinates so that the boundary is a graph $x_n = \lambda(x)$. This does not change either of the integrals in the divergence theorem. Now we consider a function F on $\bar{\Omega} \cap (V' \times (a, b)) = \{\left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right) \mid z \leq \lambda(x)\}$. We may assume that F and F' are zero on $\partial V' \times (a, b)$ and $V' \times \{a\}$. Suppose $1 \leq i < n$ and consider the function

$$x \mapsto \int_a^{\lambda(x)} F_i(x, z) \, \mathrm{d}z.$$

It vanishes for $x \in \partial V'$ as does its derivative

$$\frac{\partial}{\partial x_i} \int_a^{\lambda(x)} F_i(x, z) \, \mathrm{d}z = \frac{\partial \lambda(x)}{\partial x_i} F_i(x, \lambda(x)) + \int_a^{\lambda(x)} \frac{\partial F_i(x, z)}{\partial x_i} \, \mathrm{d}z.$$

Applying the same argument as in the first case, we see that the integral of ∂_i-derivative over V' vanishes. Therefore

$$\begin{aligned} \int_{\bar{\Omega} \cap (V' \times (a,b))} \frac{\partial F_i(x, z)}{\partial x_i} \, \mathrm{d}\mu &= \int_{V'} \int_a^{\lambda(x)} \frac{\partial F_i(x, z)}{\partial x_i} \, \mathrm{d}z \, \mathrm{d}^{\mathrm{n}-1} x = - \int_{V'} \frac{\partial \lambda(x)}{\partial x_i} F_i(x, \lambda(x)) \, \mathrm{d}^{\mathrm{n}-1} x \\ &= \int_{V'} F_i(x, \lambda(x)) N_i \sqrt{1 + |\nabla \lambda|^2} \, \mathrm{d}^{\mathrm{n}-1} x = \int_{A'} F_i \, N_i \, \mathrm{d}\sigma. \end{aligned}$$

Note that the signs required us to use the outward-pointing normal, which in this case means that the last component of the vector N is positive.

For the case $i = n$, we can just use the fundamental theorem of calculus on the inner integral

$$\begin{aligned} \int_{\bar{\Omega} \cap (V' \times (a,b))} \frac{\partial F_n(x, z)}{\partial x_n} \, \mathrm{d}\mu &= \int_{V'} \int_a^{\lambda(x)} \frac{\partial F_n(x, z)}{\partial x_n} \, \mathrm{d}z \, \mathrm{d}^{n-1} x = \int_{V'} F_n(x, \lambda(x)) \, \mathrm{d}^{n-1} x \\ &= \int_{V'} F_n(x, \lambda(x)) N_n \sqrt{1 + |\nabla \lambda|^2} \, \mathrm{d}^{n-1} x = \int_{A'} F_n \, N_n \, \mathrm{d}\sigma \end{aligned}$$

Summing these terms together proves the theorem. □

We consider now some special cases of the theorem that occur over and over in practice. For a scalar valued function f the divergence theorem implies for all $i = 1, \ldots, n$

$$\int_{\Omega} \partial_i f \, \mathrm{d}\mu = \int_{\partial\Omega} f N_i \, \mathrm{d}\sigma$$

For two functions f and g whose product vanishes on the boundary $\partial\Omega$ and satisfies the corresponding assumptions of the divergence theorem we obtain by the Leibniz rule

$$\int_\Omega f\partial_i g \,\mathrm{d}\mu = -\int_\Omega g\partial_i f \,\mathrm{d}\mu \qquad \text{for all } i = 1, \ldots, n.$$

This is called integration by parts. Inductively we get for any multi-index γ

$$\int_\Omega f\partial^\gamma g \,\mathrm{d}\mu = (-1)^{|\gamma|}\int_\Omega g\partial^\gamma f \,\mathrm{d}\mu.$$

As a second application of the divergence theorem we can generalise the idea of the scalar conservation law to vector-valued functions. For any continuously differentiable function $F : \mathbb{R} \to \mathbb{R}^n$ we call

$$\dot{u}(x,t) + \nabla \cdot F(u(x,t)) = \dot{u}(x,t) + F'(u(x,t)) \cdot \nabla u(x,t) = 0$$

a conservation law. For open and bounded $\Omega \subset \mathbb{R}^n$ with $n-1$-dimensional submanifold $\partial\Omega$ of $\mathbb{R}^n$ we obtain

$$\frac{d}{dt}\int_\Omega u(x,t)\,\mathrm{d}^\mathrm{n}x = \int_\Omega \dot{u}(x,t)\,\mathrm{d}^\mathrm{n}x = -\int_\Omega \nabla\cdot F(u(x,t))\,\mathrm{d}^\mathrm{n}x = -\int_{\partial\Omega} F(u(x,t))\cdot N(x)\,\mathrm{d}\sigma(x).$$

This is the meaning of a conservation law: the change of the integral of $u(\cdot,t)$ over $\Omega \subset \mathbb{R}^n$ is equal to the integral of the flux $-F(u(\cdot,t)) \cdot N$ through the boundary $\partial\Omega$.

This idea also gives the following cute trick to calculate the surface area of a ball in relation to its volume. Let the volume of the n-dimensional unit ball be ω_n. By scaling, the volume of the ball $B(0,r)$ is $\omega_n r^n$. Let $\sigma_n(r)$ denote the area of $\partial B(0,r) \subset \mathbb{R}^n$. The divergence of $x \mapsto x$ is n, so by the divergence theorem we have

$$n\omega_n r^n = \int_{B(0,r)} \nabla \cdot x \,\mathrm{d}\mu = \int_{\partial B(0,r)} x \cdot N(x)\,\mathrm{d}\sigma(x) = \int_{\partial B(0,r)} x \cdot \tfrac{x}{|x|}\,\mathrm{d}\sigma(x) = r\sigma_n(r).$$

In summary $\sigma_n(r) = n\omega_n r^{n-1}$.

2.6 Distributions

For the transport equation we developed a solution that also seems to make sense when it is not differentiable. For the scalar conservation law we saw that there were in some situations no solutions, except if we generalised the notion of solution to include discontinuous functions. The lesson we draw from these examples is that the existence and uniqueness of solutions depends on the notion of solution we use. In order to say that these solutions solve the PDE, clearly all partial derivatives of a solution which occur in the partial differential equation have to exist. The trick is to come up with a new notion of partial derivative and interpret the PDE to be about these new derivatives.

In this section we introduce generalised functions (called distributions) and a corresponding notion of differentiation. This notion is 'backwards compatible': if a differentiable function is considered as a distribution, the two types of derivatives are equal. Remarkably distributions can always be differentiated and indeed they can be differentiated infinitely many times. For this achievement we have to pay a price: these distributions cannot be multiplied with each other in general. Linear partial differential equations extend to well defined equations on such distributions. Distributions solving the linear partial differential equations are called *weak solutions* or solutions in the sense of distributions. There exist other notions of weak solutions which also apply to non-linear partial differential equations. The most prominent example is the notion of a Sobolev function, which are introduced in the course "Partial Differential Equations", the sequel to this course. But Sobolev functions can be understood as a special type of distribution, so even if one is interested in Sobolev functions it is helpful to start with distributions.

First we need to define a special class of very well behaved functions. The support $\operatorname{supp} f$ of a function f is the closure of $\{x \mid f(x) \neq 0\}$. On an open set $\Omega \subseteq \mathbb{R}^n$ let $C_0^\infty(\Omega)$ denote the algebra of smooth functions whose support is a compact subset of Ω. We call these *test functions* and say they have compact support in Ω, symbolically $\operatorname{supp} f \Subset \Omega$. There is a technical matter to discuss at this point. The set of test functions should be given a different topology than the norm topology of $C^\infty(\Omega)$ with the supremum norm, but this other topology is tricky and not directly important to this course. Instead, let us give the criterion for when a sequence of test functions converges: $f_n \to f$ if there is a compact subset $K \subset \Omega$ such that the supports of every f_n and f are contained in K and that $\partial^\alpha f_n$ converges to $\partial^\alpha f$ in the supremum norm on K for every multi-index α (including $\alpha = 0$). We also use the notation $\mathcal{D}(\Omega)$ for the set of test functions equipped with this topology.

Within the set of test functions there are a special families that we will often use called a *mollifier* or *approximate identities*. This is a family of non-negative test functions $(\lambda_\epsilon)_{\epsilon>0}$ with $\operatorname{supp} \lambda_\epsilon = \overline{B(0,\epsilon)}$ and $\int \lambda_\epsilon \, d\mu = 1$. We construct a prototype: the function

$$\lambda(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

is a smooth function on $\mathbb{R}^n$, has support $\overline{B(0,1)}$, and is non-negative. By the way, this example shows that test functions actually exist. We can choose the constant C such that its integral is 1. By rescaling x and λ we obtain

$$\lambda_\epsilon(x) = \epsilon^{-n}\lambda(x/\epsilon),$$

which has the required properties. This particular example of a mollifier is call the standard mollifier, but for our purposes it does not matter which mollifier we use. Any such family is called an approximate identity because of the following property. Take any continuous function f on Ω and suppose $0 \in \Omega$. By continuity f is approximately equal to $f(0)$ on a sufficiently small ball $B(0, \epsilon)$. Therefore

$$\int_{\Omega} f \lambda_\epsilon \, \mathrm{d}\mu \approx \int_{B(0,\epsilon)} f(0) \lambda_\epsilon \, \mathrm{d}\mu = f(0).$$

In fact, as we will prove in the next lemma, this approximation becomes an equality in the limit $\epsilon \downarrow 0$.

Lemma 2.8. *Let $f \in C(\Omega)$ and $(\lambda_\epsilon)_{\epsilon>0}$ be a mollifier. The family of smooth functions*

$$f_\epsilon(x) := \int_{\Omega} f(y) \lambda_\epsilon(x-y) \, \mathrm{d}^{\mathrm{n}} y$$

converges uniformly on any compact subset of Ω to f as $\epsilon \downarrow \epsilon$. For smooth functions the same holds for all derivatives of f.

Proof. Choose a compact subset of Ω. There is an ϵ such that for any point x in the compact set the ball $B(x, \epsilon)$ lies in Ω. For this ϵ or smaller we have

$$|f_\epsilon(x) - f(x)| = \left| \int_{\Omega} \lambda_\epsilon(x-y)(f(y) - f(x)) \, \mathrm{d}^{\mathrm{n}} y \right| \leq \sup_{y \in B(x,\epsilon)} |f(y) - f(x)|.$$

On compact sets continuous functions are uniformly continuous. This shows the uniform convergence $\lim_{\epsilon \downarrow 0} f_\epsilon = f$.

Observe that if f is smooth, then we can compute the derivatives of f_ϵ in the following way. Choose any point $x_0 \in \Omega$ and let ϵ be small enough that $B(x_0, 2\epsilon) \subset \Omega$. Then for all points $x \in B(x_0, \epsilon)$

$$f_\epsilon(x) = \int_{B(x,\epsilon)} f(y) \lambda_\epsilon(x-y) \, \mathrm{d}^{\mathrm{n}} y = \int_{B(0,\epsilon)} f(x-z) \lambda_\epsilon(z) \, \mathrm{d}^{\mathrm{n}} z.$$

Therefore $\partial^\alpha f_\epsilon = (\partial^\alpha f)_\epsilon$ and the same convergence argument carries over to all partial derivatives of f. □

The formula we see in the definition of f_ϵ turns out to be useful. We use it to define a type of product operator on $C_0^\infty(\mathbb{R}^n)$, the convolution

$$(g * f)(x) := \int_{\mathbb{R}^n} g(x-y) f(y) \, \mathrm{d}^{\mathrm{n}} y = \int_{\mathbb{R}^n} g(z) f(x-z) \, \mathrm{d}^{\mathrm{n}} z.$$

This product is commutative and associative (Exercise). One advantage of the convolution compared to pointwise multiplication is that it behaves nicely with differentiation. There is no Leibniz rule, rather

$$\partial^\alpha (g * f) = (\partial^\alpha g) * f = g * (\partial^\alpha f).$$

Furthermore convolution is well-behaved with respect to integral norms, which is useful in more advanced theory. We can consider the simplest case, where integral of $f * g$ is the product of the integrals of f and g. This follows by noticing that the coordinate transformation $z = y - x, y = y$ is volume preserving, thus

$$\int_{\mathbb{R}^n} (f * g)(x)\,\mathrm{d}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z)g(y)\,\mathrm{d}z\,\mathrm{d}y$$
$$= \left(\int_{\mathbb{R}^n} f(z)\,\mathrm{d}z\right)\left(\int_{\mathbb{R}^n} g(y)\,\mathrm{d}y\right)$$

Finally, we include a lemma that will be necessary later

Lemma 2.9. *Suppose that f and g are rotationally symmetric about a and b respectively. This means, for example for any orthogonal transformation O that $f(a+x) = f(a+Ox)$. Then the convolution of f and g is rotationally symmetric about $a + b$.*

Proof. The proof is just a sequence of coordinate transformations. We begin with the definition and make the euclidean motion $y = Oz + b$

$$(f * g)(a + b + Ox) = \int_{\mathbb{R}^n} f(a + b + Ox - y)g(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} f(a + O(x - z))g(b + Oz)\,\mathrm{d}z.$$

It is important to see here that $\mathrm{d}y = \mathrm{d}z$ since O is orthogonal. Now we use the orthogonal properties of f and g to continue

$$= \int_{\mathbb{R}^n} f(a + x - z)g(b + z)\,\mathrm{d}z = \int_{\mathbb{R}^n} f(a + x - y' + b)g(y')\,\mathrm{d}y' = (f * g)(a + b + x).$$

□

Now it is time to introduce distributions. We have seen in the previous lemma that the operation of integrating a continuous function against a test function somehow retains the information of the function. In this spirit each $f \in L^1_{\mathrm{loc}}(\Omega)$ defines a linear map

$$F_f : \mathcal{D}(\Omega) \to \mathbb{R}, \qquad \phi \mapsto \int_\Omega f\phi\,\mathrm{d}\mu.$$

We will see that the information of f is also retained in this linear form. The idea of distributions is consider not just functions integrated against test functions, but all linear forms F acting on $\mathcal{D}(\Omega)$. Again, there is some technical problems with convergence, and for this reason we also need a type of continuity property on the linear maps. We make the following definitions.

We define for any compact subset $K \subset \Omega$ and every multi-index α the following seminorm:

$$\|\cdot\|_{K,\alpha} : C_0^\infty(\Omega) \to \mathbb{R}, \qquad \phi \mapsto \|\phi\|_{K,\alpha} := \sup_{x\in K} |\partial^\alpha \phi(x)|.$$

Definition 2.10. *On an open subset $\Omega \subseteq \mathbb{R}^n$ the space of distributions $\mathcal{D}'(\Omega)$ is defined as the vector space space of all linear maps $F : \mathcal{D}(\Omega) \to \mathbb{R}$ which are continuous with respect to the seminorms $\|\cdot\|_{K,\alpha}$; i.e. for each compact $K \subset \Omega$ there exist finitely many multi indices $\alpha_1, \dots, \alpha_M$ and constants $C_1 > 0, \dots, C_M > 0$ such that the following inequality holds for all test functions $\phi \in \mathcal{D}(\Omega)$ with compact support in K:*

$$|F(\phi)| \leq C_1\|\phi\|_{K,\alpha_1} + \ldots + C_M\|\phi\|_{K,\alpha_M}.$$

The $\mathcal{D}'$ for distributions indicates (for the correctly defined topology) that they are the dual space of $\mathcal{D}$. Concretely the continuity condition yields the following convergence property for distributions: if $\phi_n \to \phi$ in $\mathcal{D}(\Omega)$ then the values $F(\phi_n)$ converges to $F(\phi)$. Similarly, a sequence of distribution F_n converges to F if $F_n(\phi) \to F(\phi)$ for all test functions ϕ.

As previously mentioned, any $f \in L^1_{\text{loc}}(\Omega)$ defines in a canonical way a distribution F_f. Let us verify now that it really meets the definition of distribution. For any compact subset $K \subset \Omega$ and $\phi \in \mathcal{D}(\Omega)$ with support K we have

$$|F_f(\phi)| \leq \sup_{x \in K} |\phi(x)| \|f\|_{L^1(K)}.$$

Let us give another example of a distribution, one that does not correspond to an element of $L^1_{\text{loc}}(\mathbb{R}^n)$:

$$\delta : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R} \qquad\qquad \phi \mapsto \phi(0).$$

Intuitively (and we will prove rigorously soon) any corresponding $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ would vanish on $\mathbb{R}^n \setminus \{0\}$ and would have a total integral one. Since $\{0\}$ has measure zero such a function does not exist. Distributions that come from $L^1_{\text{loc}}(\Omega)$ functions are called regular, and those that don't are non-regular. This distribution is called Dirac's δ-function. We can also show that it is the limit of the sequence of distributions corresponding to the mollifier λ_ϵ.

We now return to the question of whether the distribution F_f retains the information of f. The answer is yes.

Lemma 2.11. *(Fundamental Lemma of the Calculus of Variations) If $f \in L^1_{\text{loc}}(\Omega)$ obeys $F_f(\phi) \geq 0$ for all non-negative test functions $\phi \in C_0^\infty(\Omega)$, then f is non-negative almost everywhere. In particular the map $L^1_{\text{loc}}(\Omega) \to \mathcal{D}'(\Omega)$, $f \mapsto F_f$ is injective.*

Proof. It suffices to prove the local statement for $f \in L^1(\Omega)$. We extend f to $\mathbb{R}^n$ by setting f on $\mathbb{R}^n \setminus \Omega$ equal to zero. The extended function is also denoted by f and belongs to

$f \in L^1(\mathbb{R}^n)$. For a mollifier $(\lambda_\epsilon)_{\epsilon>0}$ we have

$$\|\lambda_\epsilon * f - f\|_1 = \int_{\mathbb{R}^n} \left| \int_{B(0,\epsilon)} \lambda_\epsilon(y) f(x-y)\, \mathrm{d^n} y - f(x) \right| \mathrm{d^n} x$$
$$\leq \int_{B(0,\epsilon)} \int_{\mathbb{R}^n} \lambda_\epsilon(y) |f(x-y) - f(x)|\, \mathrm{d^n} x\, \mathrm{d^n} y \leq \sup_{y \in B(0,\epsilon)} \|f(\cdot - y) - f\|_1.$$

If f is the characteristic functions of a rectangle, then the supremum on the right hand side converges to zero for $\epsilon \downarrow 0$. Due to the triangle inequality the same holds for step functions, i.e. finite linear combinations of such functions. Since step functions are dense in $L^1(\mathbb{R}^n)$ for each $f \in L^1(\mathbb{R}^n)$ this supremum becomes arbitrary small for sufficiently small ϵ. Hence the family of functions $(\lambda_\epsilon * f)_{\epsilon>0}$ converges in $L^1(\mathbb{R}^n)$ in the limit $\epsilon \downarrow 0$ to f.

Moreover, the functions $\lambda_\epsilon * f$ are non-negative. This is because the mollifiers are non-negative and we can write the convolution as the action of F_f on a test function

$$(\lambda_\epsilon * f)(x) = \int_{\mathbb{R}^n} \lambda_\epsilon(x-y) f(y)\, \mathrm{d^n} y = F_f(\lambda_\epsilon(x - \cdot)) \geq 0$$

using the assumption on F_f.

So it remains to show that a limit in L^1 of a sequence of non-negative functions is also non-negative. In particular there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ which converges to zero, with $\|f_n - f\|_1 \leq 2^{-n}$ for all $n \in \mathbb{N}$ for $f_n = \lambda_{\epsilon_n} * f$. This ensures that the series $\sum_{n\in\mathbb{N}} |f_n - f|$ converges in $L^1(\mathbb{R}^n)$. So for almost every point x the series $\sum_{n\in\mathbb{N}} |f_n(x) - f(x)|$ is finite, and in particular the tail of the series converges to zero. In other words $\lim_{n\to\infty} f_n(x) = f(x)$. This indeed shows that f is a.e. non-negative.

In particular, if f belongs to the kernel of $f \mapsto F_f$, then both f and $-f$ are almost everywhere non-negative. So f vanishes almost everywhere. □

Two definitions for functions carry over naturally to distributions. If $\Omega' \subset \Omega$ then every test function on Ω' extends to a test function on Ω. In this way we can think of any distribution on Ω as a distribution on Ω', which we call the restriction. For regular distributions, this is really the restriction of functions. Using restriction we can give a definition of support. The complement of the support of a distribution is the union of all sets on which the restriction vanishes. In symbols

$$(\operatorname{supp} F)^c = \bigcup \{\Omega' \subset \Omega \mid F(\phi) = 0 \quad \forall \phi \in \mathcal{D}(\Omega')\}.$$

The support of the delta distribution is $\{0\}$, and the support of the distribution of a continuous function is its support in the normal sense.

We want to define as many operations on distributions as possible, such that they extend operations on functions. Restriction and support are two examples where this is clear.

The general strategy for making such definitions is to compare F_f to F_{Af} where A is the operation. If we can write the relation in a way that only depends on the distribution and not directly on the function, then it is suitable to make a generalised definitions. Let us consider the case of multiplication by a smooth function $g \in C^\infty(\Omega)$. Then for a regular distribution

$$F_{gf}(\phi) = \int_\Omega (gf)\phi = \int_\Omega f(g\phi) = F_f(g\phi).$$

The product of a distribution with a function $g \in C^\infty(\Omega)$ is defined as

$$gF : \mathcal{D}(\Omega) \to \mathbb{R}, \qquad \phi \mapsto F(g\phi).$$

This product makes the embedding $C^\infty(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ to a homomorphism of modules over the algebra $C^\infty(\Omega)$. However, even the product of a distribution with a continuous non-smooth functions is not defined.

So we come to the most important operation on distributions. If f has a derivative, then by integration by parts we obtain

$$F_{\partial_i f} = \int_\Omega \partial_i f \phi \, \mathrm{d^n}x = -\int_\Omega f \partial_i \phi \, \mathrm{d^n}x = -F_f(\partial_i \phi).$$

Consequently for any distribution $F \in \mathcal{D}'(\Omega)$ we define the partial derivatives as

$$\partial_i F : \mathcal{D}(\Omega) \to \mathbb{R}, \qquad \phi \mapsto -F(\partial_i \phi).$$

Here we see the advantage of choosing smooth test functions: test functions are always differentiable and so distributions have infinitely many derivatives. These two operations we have just defined, multiplication with a smooth function and partial differentiation, define new distributions. Clearly these new distributions are linear. We should check that they also obey the continuity condition, but we will skip this formality.

We also want to extend convolution to distributions. In order to extend it to a product between a smooth function and a distribution we calculate:

$$\begin{aligned} F_{g*f}(\phi) &= \int_{\mathbb{R}^n} (g * f)\phi \, \mathrm{d^n}x = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(x-y) f(y) \phi(x) \, \mathrm{d^n}y \, \mathrm{d^n}x \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \phi(x) g(x-y) \, \mathrm{d^n}x \right) f(y) \, \mathrm{d^n}y = F_f(\phi * \mathsf{P}g), \end{aligned}$$

where $(\mathsf{P}g)(z) := g(-z)$ is the point-reflection operator. Therefore we define for $g \in C_0^\infty(\mathbb{R}^n)$ and $F \in \mathcal{D}'(\mathbb{R}^n)$

$$g * F : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}, \qquad \phi \mapsto F(\phi * \mathsf{P}g).$$

Not only is this a well-defined distribution, the result of convolution is in fact always a regular distribution that corresponds to a smooth function!

Lemma 2.12. *The convolution $g * F$ of a test function $g \in C_0^\infty(\mathbb{R}^n)$ with a distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $C^\infty(\mathbb{R}^n)$. It is the function*

$$g * F : \mathbb{R}^n \to \mathbb{R}, \qquad x \mapsto F(\mathsf{T}_x \mathsf{P} g)$$

where $(\mathsf{T}_x\phi)(y) := \phi(y-x)$ is the translation operator. The support of $g * F$ is contained in the pointwise sum $\mathrm{supp}(g) + \mathrm{supp}(F)$.

Proof. First we show that the function defined in the lemma exists and is smooth. The support of $\mathsf{T}_x\mathsf{P}g$ is $\{y \in \mathbb{R}^n \mid x - y \in \mathrm{supp}(g)\} = x - \mathrm{supp}(g)$. Hence for every x the value $F(\mathsf{T}_x\mathsf{P}g)$ is well defined for $F \in \mathcal{D}'(\Omega)$. Since continuous functions are uniformly continuous on compact sets, the map $x \mapsto \mathsf{T}_x\mathsf{P}g$ is continuous with respect to the seminorms $\|\cdot\|_{K,0}$. Furthermore, the same holds for the seminorms $\|\cdot\|_{K,\alpha}$ since $\frac{\mathsf{T}_{x+\epsilon h} - \mathsf{T}_x}{\epsilon} g = \mathsf{T}_x \frac{\mathsf{T}_{\epsilon h} - \mathbb{1}}{\epsilon} g$ converges in the limit $\epsilon \to 0$ for all $g \in C_0^\infty(\mathbb{R}^n)$ uniformly on $\mathbb{R}^n$ to $\mathsf{T}_x\left(\sum_{i=1}^n -h_i\partial_i g\right)$. This shows $x \mapsto F(\mathsf{T}_x\mathsf{P}g) \in C^\infty(\mathbb{R}^n)$ for $F \in \mathcal{D}'(\mathbb{R}^n)$.

Next we show this smooth function corresponds to the distribution $g * F$ we defined immediately before the lemma. For any $\phi \in \mathcal{D}(\mathbb{R}^n)$ appropriate Riemann sums define a sequence of finite linear combinations of functions in $\{\mathsf{T}_x\mathsf{P}g \in C_0^\infty(\mathbb{R}^n) \mid x \in \mathrm{supp}(\phi)\}$, which converges with respect to $\|\cdot\|_{K,\alpha}$ to $\int_{\mathbb{R}^n} \mathsf{T}_x\mathsf{P}g\phi(x)\,\mathrm{d}^n x$. Hence the linearity and continuity of F gives

$$\int_{\mathbb{R}^n} (g * F)(x)\phi(x)\,\mathrm{d}^n x = \int_{\mathbb{R}^n} F(\mathsf{T}_x\mathsf{P}g)\phi(x)\,\mathrm{d}^n x = F\left(\int_{\mathbb{R}^n} \mathsf{T}_x\mathsf{P}g\phi(x)\,\mathrm{d}^n x\right) = F(\mathsf{P}g * \phi).$$

Finally, we consider the support. If $F(\mathsf{T}_x\mathsf{P}g) \neq 0$, then $g(x-y) \neq 0$ for an element $y \in \mathrm{supp}\, F$. Hence $x = y + (x-y) \subset \mathrm{supp}\, F + \mathrm{supp}\, g$ and $\mathrm{supp}(x \mapsto F(\mathsf{T}_x\mathsf{P}g)) \subset \mathrm{supp}\, F + \mathrm{supp}\, g$. □

This Lemma implies that even the convolution of a distribution $F \in \mathcal{D}'(\mathbb{R}^n)$ with a distribution $G \in \mathcal{D}'(\mathbb{R}^n)$ with compact support $\mathrm{supp}\, G$ is a well defined distribution:

$$F * G : \mathcal{D}(\Omega) \to \mathbb{R}, \qquad \phi \mapsto F(\phi * \mathsf{P}G) \text{ with } \qquad \mathsf{P}G(\phi) := G(\mathsf{P}\phi).$$

In particular, we can convolve any distribution with the δ-distribution. Remarkably this returns the same distribution, i.e. $F * \delta = F$ (Exercise). We say that δ is the identity element or neutral element of convolution.

Further details of the theory of distributions can be found in the short and lucid first chapter of the book of Lars Hörmander: “Linear Partial Differential Operators”.

Chapter 3

Laplace Equation

One of the most important PDEs is the Laplace equation

$$\triangle u = \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.$$

The corresponding inhomogeneous PDE is Poisson's equation

$$-\triangle u = f.$$

Both equations are linear PDEs of second order with the unknown function $u : \mathbb{R}^n \to \mathbb{R}$. A function that solves Laplace's equation is called harmonic. As is typical with linear inhomogeneous equations, the sum of a solution of Poisson's equation and a harmonic function is again a solution to Poisson's equation. These equations show up in many situations. In physics they describe for example the potential of an electric field in the vacuum with some distribution of charges f.

3.1 Fundamental Solution

The Laplace equation is invariant with respect to all rotations and translations of the Euclidean space $\mathbb{R}^n$. Therefore we first look for solutions which are invariant with respect to all rotations. These solutions depend only on the length $r = |x| = \sqrt{x \cdot x}$ of the position vector x. For such functions $u(x) = v(r) = v(\sqrt{x \cdot x})$ we calculate:

$$\nabla_x u(x) = v'\left(\sqrt{x \cdot x}\right) \nabla_x r = v'\left(\sqrt{x \cdot x}\right) \frac{2x}{2r}.$$

Hence the Laplace equation simplifies to an ODE

$$\triangle_x u(x) = \nabla_x \cdot \nabla_x u = v''(r)\frac{x^2}{r^2} + v'(r)\frac{n}{r} - v'(r)\frac{x^2}{r^2 r} = v''(r) + \frac{n-1}{r}v'(r) = 0.$$

Let us solve this ODE:

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r} \Rightarrow \ln(v'(r)) = (1-n)\ln(r) + C \Rightarrow v(r) = \begin{cases} C'\ln(r) + C'' & \text{for } n = 2 \\ \frac{C'}{r^{n-2}} + C'' & \text{for } n \geq 3. \end{cases}$$

We see two things here. The space of solutions is two dimensional, with one solution being just the constant solution $u = C''$. The other solution is not a solution on all of $\mathbb{R}^n$ because is has a singularity at the origin. Never-the-less these are important 'solutions' to consider!

Definition 3.1. *Let $\Phi(x)$ be the following solutions of the Laplace equation:*

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\ln|x| & \text{for } n = 2 \\ \frac{1}{n(n-2)\omega_n|x|^{n-2}} & \text{for } n \geq 3. \end{cases}$$

Here ω_n denotes the volume of the unit ball $B(0,1)$ in Euclidean space $\mathbb{R}^n$. We call these fundamental solutions of the Laplace equation.

This solution lies in the space of symmetric solutions. We have chosen $C'' = 0$, which makes the solution tend to zero for large x. The constant C' is chosen in such a way that the following theorem holds:

Theorem 3.2. *For $f \in C_0^2(\mathbb{R}^n)$ a solution of Poisson's equations $-\triangle u = f$ is given by*

$$u(x) = \Phi * f = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, \mathrm{d}^n y.$$

Moreover, the distribution corresponding to the fundamental solution obeys $-\triangle F_\Phi = \delta$.

Proof. We see that the function u is twice continuously differentiable since f is twice continuously differentiable and because it has compact support we can differentiate under the integral sign. We calculate

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)\, \mathrm{d}^n y.$$

In particular, $\triangle u(x) = \int_{\mathbb{R}^n} \Phi(y) \triangle_x f(x-y)\, \mathrm{d}y$. We decompose this integral in the sum of an integral nearby the singularity of Φ and an integral away from this singularity:

$$\begin{aligned} \triangle u(x) = &\int_{B(0,\epsilon)} \Phi(y)\triangle_x f(x-y)\, \mathrm{d}y &&+ \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y)\triangle_x f(x-y)\, \mathrm{d}y \\ = &I_\epsilon &&+ J_\epsilon. \end{aligned}$$

We use $\int r \ln r dr = \frac{r^2}{2}(\ln r - \frac{1}{2})$ and $\int r dr = \frac{r^2}{2}$ and estimate the first integral for $\epsilon \downarrow 0$:

$$|I_\epsilon| \leq \|\triangle_x f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\epsilon)} |\Phi(y)|\, \mathrm{d}y \leq \begin{cases} C\epsilon^2(|\ln\epsilon| + 1) & (n = 2) \\ C\epsilon^2 & (n \geq 3). \end{cases}$$

In the J_ϵ integral, because $\triangle$ is second order, we can change $\triangle_x f(x-y)$ to $\triangle_y[f(x-y)]$ without changing signs. Then integration by parts yields

$$\begin{aligned} J_\epsilon &= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y) \nabla_y \cdot \nabla_y [f(x-y)] \,\mathrm{d}y \\ &= -\int_{\mathbb{R}^n \setminus B(0,\epsilon)} \nabla_y \Phi(y) \cdot \nabla_y [f(x-y)] \,\mathrm{d}y \quad + \int_{\partial B(0,\epsilon)} \Phi(y) \nabla_y [f(x-y)] \cdot N \,\mathrm{d}\sigma(y) \\ &= K_\epsilon \quad + L_\epsilon. \end{aligned}$$

We are able to apply integration by parts because f has compact support; we can restrict $\mathbb{R}^n$ to some large ball without changing the integral. The second term converges in the limit $\epsilon \downarrow 0$ to zero:

$$|L_\epsilon| \le |\nabla f|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\epsilon)} |\Phi(y)| \,\mathrm{d}\sigma(y) \le \begin{cases} C\epsilon |\ln \epsilon| & (n=2) \\ C\epsilon & (n \ge 3). \end{cases}$$

Another integration by parts of the first term yields

$$\begin{aligned} K_\epsilon &= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \triangle_y \Phi(y) f(x-y) \,\mathrm{d}y - \int_{\partial B(0,\epsilon)} \nabla_y \Phi(y) f(x-y) \cdot N \,\mathrm{d}\sigma(y) \\ &= -\int_{\partial B(0,\epsilon)} \nabla_y \Phi(y) f(x-y) \cdot N \,\mathrm{d}\sigma(y). \end{aligned}$$

Here we used that Φ is harmonic for $y \ne 0$. The gradient of Φ is equal to $\nabla\Phi(y) = -\frac{1}{n\omega_n} \frac{y}{|y|^n}$. The outer normal N of $\mathbb{R}^n \setminus B(0,\epsilon)$ on $\partial B(0,\epsilon)$ points towards the origin and is given by the expression $-\frac{y}{|y|}$. Together $\nabla_y \Phi(y) \cdot N = \frac{1}{n\omega_n} \frac{1}{|y|^{n-1}}$ As we will prove rigorously in Lemma 3.3, the limit of K_ϵ as $\epsilon \to 0$ is $-f(x)$. We can understand this intuitively by observing that for ϵ small and $y \in \partial B(0,\epsilon)$ by continuity $f(x-y) \approx f(x)$. Therefore

$$K_\epsilon \approx -\int_{\partial B(0,\epsilon)} f(x) \frac{1}{n\omega_n} \frac{1}{|\epsilon|^{n-1}} \,\mathrm{d}\sigma(y) = -f(x) \frac{1}{n\omega_n |\epsilon|^{n-1}} \int_{\partial B(0,\epsilon)} 1 \,\mathrm{d}\sigma(y) = -f(x).$$

Putting these three limits together

$$\triangle u(x) = I_\epsilon + K_\epsilon + L_\epsilon \to 0 - f(x) + 0.$$

Because the left hand side is independent of ϵ, we conclude that it must have been equal to $-f(x)$ all along.

It remains to prove the claim about distributions. For any test function φ we have per the definition of distribution derivative

$$(\triangle F_\Phi)(\varphi) = F_\Phi(\triangle \varphi) = \int_{\mathbb{R}^n} \Phi(y) \triangle \varphi(y) \,\mathrm{d}^{\mathrm{n}} y.$$

But then we can see this as the calculation above with $\varphi(y) = f(0-y)$. The conclusion is that the value of the integral is $-\varphi(0)$. Moving the minus sign around we arrive at $-\triangle F_\Phi(\varphi) = \varphi(0)$. But this is the definition of the delta distribution. □

In general, a fundamental solution of a constant coefficient linear PDE $Lu = f$ has the property that $L\Phi = \delta$ in the sense of distribution. We make these assumptions on L so that L is just the real-linear combination of partial derivatives, and so interacts well with convolution. In particular, if we apply L to the convolution of f and the fundamental solution

$$L(\Phi * f) = (L\Phi) * f = \delta * f = f.$$

This shows that the convolution $\Phi * f$ solves the inhomogeneous PDE as long as it is well defined and the derivative rule for convolutions holds.

Fundamental solutions are not usually unique however. Consider the present case of the Laplace equation. If we have any harmonic function v then $\triangle(\Phi + v) = \triangle\Phi + \triangle v = \delta + 0$ shows that $\Phi + v$ is also a fundamental solution. The difference between two fundamental solutions solves the Laplace equation, so this is the only possibility for other fundamental solutions. Different fundamental solutions can produce different solutions to the PDE, though the example of $f = 0$ shows that this is not necessarily the case. We shall see that the fundamental solution we have chosen is the only one that vanishes at infinity, which makes it in some sense the best one.

The difference between the first and second claim of the theorem is the assumption of regularity of f: twice continuously differentiable or smooth respectively. In fact it is possible to generalise this theorem further: the convolution of f with Φ is defined for continuous functions $f \in L^1(\mathbb{R}^n)$ and belongs to $L^1(\mathbb{R}^n)$. In this case the result of the convolution may not be differentiable but it is a solution of Poisson's equation in the sense of distributions. However, if one assumes that f is Lipschitz continuous and belongs to $L^1(\mathbb{R}^n)$ then u is twice differentiable (in the usual sense) and solves the PDE. This situation is typical of the delicate questions of regularity of the solution.

3.2 Mean Value Property

In the previous section we constructed a solution to the inhomogeneous equation. Any other solution must differ from the constructed one by a harmonic function. We should therefore understand harmonic functions in order to understand the space of solutions. In this section we shall prove the following property of a harmonic function u on an open domain $\Omega \subset \mathbb{R}^n$: the value $u(x)$ of u at the center of any ball $B(x, r)$ with compact closure in Ω is equal to the mean of u on the boundary of the ball. Conversely, if this holds for all balls with compact closure in Ω, then u is harmonic. This relation is called mean value property and has many important consequences.

Let us introduce some notation. Given a function u let

$$\mathcal{S}[u](x, r) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y)\,\mathrm{d}\sigma(y) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} u(x + rz)\,\mathrm{d}\sigma(z)$$

be its *spherical mean*. Here ω_n denotes the volume of the unit ball in Euclidean space $\mathbb{R}^n$. We write $\mathcal{S}(r)$ when the function and center point are clear. The mean of u on the ball $B(x,r)$ is the mean over $r' \in [0,r]$ of the spherical means of u on $\partial B(x,r')$. Many statements can therefore be made either in terms of globular means or spherical means. The exact relation between them

$$\int_{B(x,r)} u \,\mathrm{d}\mu = \int_0^r \left(\int_{\partial B(x,s)} u \,\mathrm{d}\sigma \right) \mathrm{d}s.$$

will be proven in an exercise.

The spherical mean, and means generally, have several nice properties. First note that the normalisation constant in the definition ensures that $\mathcal{S}[1] = 1$ and likewise for any other constant. The mean is real-linear in the function: $\mathcal{S}[au + bv] = a\mathcal{S}[u] + b\mathcal{S}[v]$, which just follows from linearity of the integral. Likewise it follows from the monotonicity of the integral that if $u \leq v$ then $\mathcal{S}[u] \leq \mathcal{S}[v]$. From these basic properties follows continuity at the center:

Lemma 3.3. *If u is a continuous function then* $\lim_{r\downarrow 0} S[u](x,r) = u(x)$.

Proof. By the definition of continuity for all $\varepsilon > 0$ there is a radius δ such that for all points $y \in B(x,\delta)$ we know $|u(y) - u(x)| < \varepsilon$. For any $r < \delta$ it follows that

$$|\mathcal{S}[u] - u(x)| = |\mathcal{S}[u] - \mathcal{S}[u(x)]| = |\mathcal{S}[u - u(x)]| \leq \mathcal{S}[|u - u(x)|] < \mathcal{S}[\varepsilon] = \varepsilon.$$

But this is the definition that $\lim_{r\downarrow 0} S[u](x,r) = u(x)$. □

Particularly important is the relationship between the spherical mean and the Laplacian of u. Differentiating the spherical mean with respect to the radius and using the divergence theorem gives

$$\begin{aligned}\frac{\partial}{\partial r}\mathcal{S}(r) &= \frac{1}{n\omega_n}\int_{\partial B(0,1)} \frac{d}{dr}(u(x+rz))\,\mathrm{d}\sigma(z) = \frac{1}{n\omega_n}\int_{\partial B(0,1)} \nabla u(x+rz)\cdot z \,\mathrm{d}\sigma(z) \\ &= \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(x,r)} \nabla u(y)\cdot N \,\mathrm{d}\sigma(y) = \frac{1}{n\omega_n r^{n-1}}\int_{B(x,r)} \triangle u \,\mathrm{d}\mu. \qquad (3.4)\end{aligned}$$

Therefore if u is harmonic then $\mathcal{S}(r)$ is constant. With these important properties of means prepared, we are ready to fully prove our claim.

Theorem 3.5 (Mean Value Property)**.** *Let $u \in C(\Omega)$ on an open domain $\Omega \subset \mathbb{R}^n$. We say that u has the mean value property if*

$$u(x) = \mathcal{S}[u](x,r) = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(x,r)} u(y)\,\mathrm{d}\sigma(y)$$

for all balls with $\overline{B(x,r)} \subset \Omega$. A twice continuously differentiable function $u \in C^2(\Omega)$ has the mean value property if and only if it is harmonic. Additionally, the same result holds if globular means are used in place of spherical means.

Proof. We have just calculated that if u is harmonic then $\mathcal{S}(r)$ is constant. From the previous lemma we then conclude that $\mathcal{S}(r) = u(x)$ for all applicable r. Conversely, if $\triangle u(x) \neq 0$, then by the continuity of $\triangle u$ there is a ball $B(x,r)$ where $\triangle u$ is strictly positive (or negative). For this ball and any ball contained in it the right hand side of equation (3.4) is strictly positive (or negative) and the spherical mean is strictly monotonic. Therefore it is not constant.

To show the statement about globular means we need to use that the integral over a ball is an iterated integral over spheres

$$\frac{1}{\omega_n r^n}\int_{B(x,r)} u \, \mathrm{d}\mu = \frac{n}{r^n}\int_0^r \frac{s^{n-1}}{n\omega_n s^{n-1}}\int_{\partial B(x,s)} u \, \mathrm{d}\sigma \, \mathrm{d}s = \frac{n}{r^n}\int_0^r s^{n-1}\mathcal{S}(s)\,\mathrm{d}s.$$

Thus if $\mathcal{S}$ is constant and equal to $u(x)$, so is the globular mean. If the globular mean is constant and equal to $u(x)$ then we differentiate both sides with respect to r

$$0 = -\frac{n^2}{r^{n+1}}\int_0^r s^{n-1}\mathcal{S}(s)\,\mathrm{d}s + \frac{n}{r^n}r^{n-1}\mathcal{S}(r) = -\frac{n}{r}u(x) + \frac{n}{r}\mathcal{S}(r).$$

Therefore $\mathcal{S}(r) = u(x)$ too. □

Keeping with our theme of distributions, we might wonder how we can reinterpret the mean value property for distributions. We saw in Section 2.6 that the value of a continuous function can be determined by integrating against a mollifier and taking the limit. The idea is that the support of the family decreases to a single point in the limit. In the same way we will now define what might be called 'sphere mollifiers' in contrast to 'point mollifiers':

$$\Lambda_{r,\epsilon}(y) := \frac{\lambda_\varepsilon(|y-x|-r)}{n\omega_n |y-x|^{n-1}}$$

for a mollifier λ_ϵ. If one looks at the graph of this function, one will see that it is bump concentrated near the sphere $\partial B(x,r)$. Suppose that u is a continuous function.

$$\int_{\mathbb{R}^n} u\Lambda_{r,\epsilon}\,\mathrm{d}\mu = \int_0^\infty\int_{\partial B(x,s)} u(y)\frac{\lambda_\varepsilon(s-r)}{n\omega_n s^{n-1}}\,\mathrm{d}\sigma(y)\,\mathrm{d}s = \int_0^\infty \mathcal{S}[u](x,s)\lambda_\varepsilon(s-r)\,\mathrm{d}s \to \mathcal{S}[u](x,r)$$

as $\epsilon \downarrow 0$, using the fact that the spherical mean is continuous in the radius and Lemma 2.8.

We might try to define the spherical mean of a distribution F to be the limit of $F(\Lambda_{r,\epsilon})$. Unfortunately this limit does not in general exist, just as one cannot define the pointwise value of a distribution to be limit of it applied to a mollifier (we have seen that $\delta(\lambda_\epsilon)$ has no limit, for example). If the distribution is harmonic however, the limit does turn out to exist. But moreover, the spherical mean is constant, so it is not even necessary to take a limit. Indeed if $u \in C^2(\Omega)$ is harmonic, then for $B(x,r) \subset \Omega$ and $\psi \in C_0^\infty((0,r))$ we have

$$\int_{B(x,r)} u(y)\frac{\psi(|y-x|)}{n\omega_n|y-x|^{n-1}}\,\mathrm{d}^n y = \int_0^r \frac{\psi(s)}{ns^{n-1}\omega_n}\int_{\partial B(x,s)} u(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}s = \left(\int_0^r \psi(s)\,\mathrm{d}s\right)u(x).$$

All that is necessary to make the integral on the left, which we are treating as a kind of generalised spherical mean, equal to the value at the center $u(x)$ is that $\int \psi = 1$. Notice that the expression on the left is linear in ψ, so if we take two radial profiles ψ_1 and ψ_2 then they give the same value if their integrals are equal, i.e. if $\int(\psi_1 - \psi_2) = 0$. This is the insight we use to generalise the mean value property to distributions.

Theorem 3.6 (Weak Mean Value Property)**.** *Let $U \in \mathcal{D}'(\Omega)$ be a distribution on an open domain $\Omega \subset \mathbb{R}^n$. It is called harmonic if $\triangle U = 0$ in the sense of distributions. We say that U has the weak mean value property if for each ball $B(x,r)$ with $B(x,r) \subset \Omega$ and each $\psi \in C_0^\infty((0,r))$ with $\int \psi \,\mathrm{d}\mu = 0$ the distribution U vanishes on the following test function:*

$$\tilde{\psi} \in C_0^\infty(\Omega), \quad y \mapsto \tilde{\psi}(y) = \frac{\psi(|y-x|)}{n\omega_n|y-x|^{n-1}} \quad \textit{with} \quad \operatorname{supp}\tilde{\psi} \subset B(x,r) \subset \Omega.$$

(a) If U is a harmonic distribution then it has the weak mean value property.

(b) Suppose $U = F_u$ for a continuous function $u \in C(\Omega)$. Then U has the weak mean value property if and only if u has the mean value property.

Proof. For the first statement it suffices to show that there exists a test function $g \in C_0^\infty(\Omega)$ with $\triangle g = \tilde{\psi}$ because then $U(\tilde{\psi}) = U(\triangle g) = (\triangle U)(g) = 0$. By the assumption on ψ that the total integral is zero we can define a test function $\Psi \in C_0^\infty((0,r))$ through $\Psi(s) = \int_0^s \psi$ with $\Psi' = \psi$. Then we define

$$g(y) = v(|y-x|) \quad \text{with} \quad v(t) = \int_r^t \frac{\Psi(s)}{n\omega_n s^{n-1}}\,\mathrm{d}s.$$

This function g depends only on $|y-x|$. Because one end of the integral is set at r and Ψ has compact support, g has compact support in $B(x,r) \subset \Omega$. Similarly it is constant on $B(x,\epsilon)$ for some $\epsilon > 0$. For y near x therefore, $\triangle_y g = 0 = \tilde{\psi}(y)$. And for $y \neq x$ we can reuse the calculation of the Laplacian for radial function from the search for the fundamental solution:

$$\triangle_y g(y) = v''(|y-x|) + \frac{n-1}{|y-x|} v'(|y-x|)$$

This implies

$$\triangle_y g(y) = \frac{\psi(|y-x|)}{n\omega_n|y-x|^{n-1}} - \frac{(n-1)\Psi(|y-x|)}{n\omega_n|y-x|^n} + \frac{n-1}{|y-x|}\frac{\Psi(|y-x|)}{n\omega_n|y-x|^{n-1}} = \tilde{\psi}(y).$$

For (b) assume $U = F_u$ for a continuous function u. If u has the mean value property, then F_u has the weak mean value property by the equation before this lemma. Conversely, for any $B(x,r)$ with compact closure in Ω, there exists $R > r$ with $B(x,R) \subset \Omega$. We use the spherical mollifiers $\Lambda_{r,\epsilon}$ defined above. For all $0 < r_1 < r_2 < R$ and sufficiently small ϵ the mollifiers $\lambda_\epsilon(t - r_1)$ and $\lambda_\epsilon(t - r_2)$ have compact support in $(0, R)$ and total integral one. Therefore $\Lambda_{r_1,\epsilon} - \Lambda_{r_2,\epsilon}$ is a function to which the weak mean value property applies, and we conclude

$$F_u(\Lambda_{r_1,\epsilon}) = F_u(\Lambda_{r_2,\epsilon}).$$

Taking the limit as $\epsilon \downarrow 0$ tells us that $\mathcal{S}[u](x, r_1) = \mathcal{S}[u](x, r_2)$. Therefore u has the mean value property. □

The set of functions $\tilde{\psi}$ in the statement of the weak mean value property may seem a bit contrived, but in fact it is a rather natural set. They are characterised by three properties:

1. they are a smooth function of the distance $|y - x|$ of the variable y to some center $x \in \Omega$,
2. they have compact support in $\mathbb{R}^n \setminus \{x\}$ and
3. their integrals vanish.

It is clear that $\tilde{\psi}$ has the first two properties and conversely any function with property 1 and 2 can be as $\tilde{\psi}(y) = \frac{\psi(|y-x|)}{n\omega_n|y-x|^{n-1}}$ for some center x and function $\psi \in C_0^\infty((0, r))$. Property 3 follows by recognising that $u \equiv 1$ is a harmonic function and reusing the calculation before Theorem 3.6: the integral of $\tilde{\psi}$ is zero if and only if $\int_0^r \psi$ is zero.

These functions also behave well under convolution, so long as its the convolution of a 'big sphere' with a 'little sphere'. By this we mean the following. Let $\tilde{\chi}, \tilde{\psi}$ be functions that obey Properties 1 and 2, with centers a, b respectively. Further suppose that $\tilde{\chi}$ is identically zero on $B(a, R)$ and the support of $\tilde{\psi}$ lies in $B(b, r)$ for $r < R$. Then $\tilde{\chi} * \tilde{\psi}$ also obeys Property 1 and 2. Let us demonstrate this now. First, due to Lemma 2.9 we know that $\tilde{\chi} * \tilde{\psi}$ is rotationally symmetric around $a + b$. Second, the convolution has compact support in $\mathbb{R}^n$ by the addition formula for supports. It remains to show that it vanishes in a neighbourhood of $a + b$. But this too follows from the addition formula for the support of a convolution, since $a + b \notin (\mathbb{R}^n \setminus B(a, R)) + B(b, r)$. There is a final point to be made; as we discussed when we introduced convolutions, the integral of $\tilde{\chi} * \tilde{\psi}$ is the product of the integral of each function. Thus the convolution has Property 3 if and only if at least one of $\tilde{\chi}$ and $\tilde{\psi}$ have it.

Now we ready to complete the reverse implication of the weak mean value property: a distribution has the weak mean value property if and only if it is a harmonic distribution. Something stronger comes out of this proof, a famous result that is known as Weyl's lemma. For this reason we state it in the following way. The strategy of the proof is as follows: for any distribution that has the weak mean value property, we can define a function through generalised spherical means. This function is smooth, harmonic, and turns out to correspond to the distribution.

Weyl's Lemma 3.7. *On an open domain $\Omega \subset \mathbb{R}^n$ for each harmonic distribution $U \in \mathcal{D}'(\Omega)$ there exists a harmonic function $u \in C^\infty(\Omega)$ with $U = F_u$.*

Proof. Let us first define u. For all $x \in \Omega$ choose a ball $B(x, r) \subset \Omega$ and a test function $\psi \in C_0^\infty((0, r))$ with $\int_0^r \psi(s)\,\mathrm{d}s = 1$. Then we define

$$u(x) := U(\tilde{\psi}_x) \quad \text{with} \qquad \tilde{\psi}_x(y) := \frac{\psi(|y-x|)}{n\omega_n|y-x|^{n-1}}.$$

If U has the weak mean value property then this definition does not depend on the choice of r and ψ. Hence we can use in the formula for $u(x)$ the same ψ for all x in a small neighbourhood of each x_0. We recognise that u is the convolution of the test function $\tilde{\psi}_0$ with the distribution U. Due to Lemma 2.12, u is smooth.

Next we prove that the distribution F_u has the weak mean value property. How does F_u act on a test function φ? Again this is answered by Lemma 2.12, $F_u(\varphi) = U(\varphi * \mathsf{P}\tilde{\psi}_0)$. This formula simplifies a little due to $\tilde{\psi}_0 = \mathsf{P}\tilde{\psi}_0$ being a radial function. Let $\tilde{\chi}$ be any function from the definition of the weak mean value property. Then we must show that $U(\tilde{\chi} * \tilde{\psi}_0) = 0$. The trick is to use the freedom definition of u to choose a suitable $\tilde{\psi}_0$. We know that there is an $\epsilon > 0$ such that $\tilde{\chi}$ vanishes on $B(x, \epsilon)$. We can choose $\tilde{\psi}_0$ such that its support lies inside the ball $B(0, \epsilon/2)$. Then by the discussion above we know that $\tilde{\chi} * \tilde{\psi}_0$ is again a function of the form considered in the weak mean value property. Therefore $F_u(\tilde{\chi}) = U(\tilde{\chi} * \tilde{\psi}_0) = 0$. In other words F_u has the weak mean value property. It follows immediately from part (b) of the previous theorem that u has the mean value property and further by Theorem 3.5 that u is harmonic.

Finally we prove $F_u = U$. The functions $\kappa_\epsilon(t) = \lambda_{\epsilon/3}(t - 2/3\epsilon)$ have support $[\epsilon/3, \epsilon]$ and total integral 1. Thus the corresponding functions $\tilde{\kappa}_\epsilon$ are a smooth mollifiers. We again use the freedom in the choice of $\tilde{\psi}$ to see that $F_u = \tilde{\kappa}_\epsilon * U$ for every ϵ. Now Lemma 2.8 implies $F_u = U$. □

Read the proof again and note that we only use the fact that U has the weak mean value property. Therefore we have actually we have proven that any distribution U that has the weak mean value property corresponds to a smooth harmonic function. Therefore the weak solutions of the Laplace equations coincide with the strong solutions, and all solutions are smooth.

To conclude this section we show that the mean value property leads to a growth estimate.

Corollary 3.8. *Let u be a harmonic function on an open domain $\Omega \subset \mathbb{R}^n$ and $B(x, r)$ a ball with compact closure in Ω. For all multi-indices α we have the estimate*

$$|\partial^\alpha u(x)| \leq C(n, |\alpha|) r^{-|\alpha|} \|u\|_{L^\infty(\overline{B(x,r)})} \quad \text{with} \quad C(n, |\alpha|) = 2^{\frac{|\alpha|(1+|\alpha|)}{2}} n^{|\alpha|}.$$

Proof. We have just seen in Weyl's lemma that all harmonic functions are smooth and thus all partial derivatives of a harmonic function are harmonic. The mean value property and integration by parts (the divergence theorem version) yield for $i = 1, \ldots, n$

$$|\partial_i \partial^\alpha u(x)| = \left| \frac{2^n}{\omega_n r^n} \int_{B(x,r/2)} \partial_i \partial^\alpha u \, \mathrm{d}\mu \right| = \left| \frac{2^n}{\omega_n r^n} \int_{\partial B(x,r/2)} \partial^\alpha u N_i \, \mathrm{d}\sigma \right| \le \frac{2n}{r} \|\partial^\alpha u\|_{L^\infty(\partial B(x,r/2))}.$$

The inductive application gives first $C(n, 1) = 2n$, and using the induction hypothesis

$$\|\partial^\alpha u(y)\| \le 2^{|\alpha|} C(n, |\alpha|) r^{-|\alpha|} \|u\|_{L^\infty(B(x,r))} \quad \text{for all} \quad y \in \partial B(x, r/2)$$

the relation $C(n, 1 + |\alpha|) = 2^{1+|\alpha|} n C(n, |\alpha|)$. The given $C(n, |\alpha|)$ is the solution. □

Liouville's Theorem 3.9. *On $\mathbb{R}^n$ a bounded harmonic function is constant.*

Proof. The foregoing corollary shows that $|\partial_i u(x)|$ is bounded by $2n\|u\|_{L^\infty(\mathbb{R}^n)} r^{-1}$ for each $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$. In the limit $r \to \infty$ the first partial derivatives vanish identically. Therefore u is constant. □

3.3 Maximum Principle

We have already mentioned the intuition that if a harmonic function is increasing in some direction then it must decreasing in another. This would imply that a harmonic function cannot have a local extremum, and this is indeed the case. Suppose a harmonic function u has a maximum at a point x of an open connected domain $\Omega \subset \mathbb{R}^n$. The mean value property implies on all balls $B(x, r) \subset \Omega$

$$\frac{1}{r^n \omega_n} \int_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}y = \frac{1}{r^n \omega_n} \int_{B(x,r)} u(x) - u(y) \, \mathrm{d}y = 0.$$

By the fundamental lemma of the calculus of variations (or a standard argument from continuity), we must conclude that $u(y) = u(x)$ for all $y \in B(x, r)$. Hence u takes the maximum on all these balls $B(x, r) \subset \Omega$. This shows that the set $\{y \in \Omega \mid u(y) = u(x)\}$ is open. But it is also the preimage of a single value, and therefore closed. It is non-empty since by assumption u does have a maximum. By the definition of connected, this set must be all of Ω.

Strong Maximum Principle 3.10. *If a harmonic function u has on a connected open domain $\Omega \subset \mathbb{R}^n$ a maximum, then u is constant.* □

There is a more geometric proof in the case that Ω is path connected. We again begin with showing that u takes its maximum on every ball centered at x in the domain. Since Ω is path-connected every other point $y \in \Omega$ is connected with x by a continuous path

$\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$. The compact image $\gamma[0, 1]$ is covered by finitely many balls $B(\gamma(t_1), r_1), \ldots, B(\gamma(t_N), r_N) \subset \Omega$ with $0 \leq t_1 < \ldots t_N \leq 1$ and $r_1, \ldots, r_N > 0$. Supplementing the balls if necessary, we can assume that the center of each ball belongs to the previous ball. Then repeating the argument Inductively, u is constantly $u(x)$ on all these balls too and hence $u \equiv u(x)$ along γ, and on Ω since this is true for all $y \in \Omega$.

A practical consequence is the following

Weak Maximum Principle 3.11. *Let the harmonic function u on a bounded open domain $\Omega \subset \mathbb{R}^n$ extend continuously to the boundary $\partial\Omega$. The maximum of u is taken on the boundary $\partial\Omega$.*

Proof. By Heine Borel the closure $\bar{\Omega}$ is compact and the continuous function u takes on $\bar{\Omega}$ a maximum. If it does not belong to $\partial\Omega$, then u is constant on the corresponding connected component and the maximum is also taken on $\partial\Omega$. □

Since the negative of a harmonic function is harmonic the same conclusion holds for minima.

The triumph of the Maximum Principle is that it generalises to many elliptic operators, unlike the mean value property. It really goes to the heart of ellipticity.

Definition 3.12. *On an open domain $\Omega \subset \mathbb{R}^n$ an differential operator L*

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u(x)}{\partial x_i}$$

with symmetric coefficients $a_{ij} = a_{ji}$ is called elliptic, if

$$\sum_{i,j=1}^{n} a_{ij}(x) k_i k_j > 0 \quad \text{for all } x \in \Omega \text{ and all } k \in \mathbb{R}^n \setminus \{0\}.$$

If we replace a_{ij} by $\frac{1}{2}(a_{ij} + a_{ji})$, then the assumption $a_{ij} = a_{ji}$ is fulfilled. Due to the commutativity of the second derivatives this replacement does not change L.

Theorem 3.13. *Let L be an elliptic operator on a bounded open domain $\Omega \subset \mathbb{R}^n$ whose coefficients a_{ij} and b_i extend continuously and elliptic to $\partial\Omega$. Every twice differentiable solution u of $Lu \geq 0$ which extends continuously to $\partial\Omega$ takes its maximum on $\partial\Omega$.*

Proof. Let us first show that L is uniform elliptic, i.e. there exists $\lambda > 0$ with

$$\sum_{i,j=1}^{n} a_{ij}(x) k_i k_j \geq \lambda \sum_{i=1}^{n} k_i^2 \quad \text{for all } x \in \Omega \text{ and all } k \in \mathbb{R}^n.$$

The continuous function $(x,k) \mapsto \sum_{i,j=1}^n a_{ij}(x)k_ik_j$ attains on the compact set $(x,k) \in \bar{\Omega} \times S^{n-1} \subset \bar{\Omega} \times \mathbb{R}^n$ a minimum $\lambda > 0$. Hence L is uniform elliptic.

Next we use a trick to move to the case where L of the function is strictly positive. For $v(x) = \exp(\alpha x_1)$ with $\alpha > 0$ we conclude

$$Lv = \alpha(\alpha a_{11}(x) + b_1(x))v \geq \alpha(\alpha\lambda + b_1(x))v.$$

The continuous coefficients b_i are bounded on the compact set $\bar{\Omega}$. Therefore there exists $\alpha > 0$ with $Lv > 0$. By linearity of L we obtain $L(u + \epsilon v) > 0$ on Ω for all $\epsilon > 0$.

Now we show that the continuous functions $u + \epsilon v$ cannot attain a maximum on Ω even though they must attain a maximum on $\bar{\Omega}$. At any such interior maximum $x_0 \in \Omega$ the first derivative of the function $u + \epsilon v$ which is twice differentiable on Ω vanishes and the Hessian is negative semi-definite. At this point we need a little bit of linear algebra to explain the connection between the Hessian and the Laplacian. The Hessian is a real symmetric matrix, so it is diagonalizable by an orthogonal matrix O, that is $H = O^TDO$. D is a diagonal matrix whose entries are the eigenvalues of H. Because H is negative semidefinite, all the eigenvalues are negative or zero. In symbols $\frac{\partial^2(u+\epsilon v)(x_0)}{\partial x_i \partial x_j} = \sum_k O_{ki}\lambda_k O_{kj}$. The Laplacian is the trace of the Hessian. Therefore

$$\triangle u(x_0,t_0) = \mathsf{tr}H = \mathsf{tr}(O^TDO) = \mathsf{tr}(DOO^T) = \mathsf{tr}(DI) = \mathsf{tr}(D) = \sum \lambda_i \leq 0.$$

Similarly, for any elliptic operator

$$L(u+\epsilon v)(x_0) = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2(u+\epsilon v)(x_0)}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)0 = \sum_{i,j,k=1}^n a_{ij}(x)O_{ki}\lambda_k O_{kj}$$

Because the eigenvalues are non-positive, we define $B_{ki} = O_{ki}\sqrt{-\lambda_k}$. Continuing with the calculation

$$L(u+\epsilon v)(x_0) = -\sum_{k=1}^n \sum_{i,j=1}^n a_{ij}(x)B_{ki}B_{kj} \leq -\sum_{k=1}^n \lambda \sum_{i=1}^n B_{ki}^2 \leq 0,$$

and this contradicts $L(u+\epsilon v) > 0$. Therefore for all $\epsilon > 0$ the maximum of $u+\epsilon v$ belongs to the boundary. Finally, we use the following comparison between u and $u + \epsilon v$ to reach the conclusion.

$$\sup_{x\in\Omega} u(x) + \epsilon \inf_{x\in\Omega} v(x) \leq \sup_{x\in\Omega}(u(x) + \epsilon v(x)) = \max_{x\in\partial\Omega}(u(x) + \epsilon v(x)) \leq \max_{x\in\partial\Omega} u(x) + \epsilon \max_{x\in\partial\Omega} v(x).$$

Because this holds for all $\epsilon > 0$ the boundedness of v on $\bar{\Omega}$ implies the theorem. □

The negative of the functions u in the theorem obey $Lu \leq 0$ and take a minimum on the boundary. In particular, the solutions u of $Lu = 0$ take the maximum and the minimum on the boundary.

Now let us see why maximum principles are so important. We consider the following very natural boundary value problem:

Dirichlet Problem 3.14. *For a given function f on a bounded open domain $\Omega \subset \mathbb{R}^n$ and g on $\partial\Omega$ we look for a solution u of $-\triangle u = f$ on Ω which extends continuously to $\partial\Omega$ and coincides there with g.*

The condition that u extends continuously to the boundary is necessary for the boundary value problem to be meaningful. Otherwise the values on the boundary could be complete unrelated to the rest of the function. We say that a function u is m times continuously differentiable on the closure $\bar{\Omega}$ of an domain, if it is m times continuously differentiable on Ω and all partial derivatives of order at most m extend continuously to $\partial\Omega$.

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain and suppose that there are two solutions u_1 and u_2 to the Dirichlet problem for the Poisson equation with inhomogeneous term f and boundary value g. Then the difference $v := u_2 - u_1$ solves the homogeneous problem, i.e. it is harmonic, and $v \equiv 0$ on $\partial\Omega$. Therefore by the weak maximum principle we know that both the maximum and minimum of v on every connected component of Ω is 0. The only possibility is that $v \equiv 0$ on all of Ω. This shows that solutions to the Dirichlet problem are unique.

Putting this another way, we can uniquely determine a harmonic function if we know its values on the boundary of its domain. This gives us a way to understand the space of harmonic functions.

3.4 Green's Function

We just saw that the solution to the Dirichlet problem is unique, if a solution exists. In this section we try to find some conditions which ensure the existence.

First we prepare some well known formulas, which hopefully you have already proved as an exercise. In first formula we apply the Divergence Theorem to $x \mapsto v(x)\nabla u(x)$:

Green's First Formula 3.15. *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for two functions $u, v \in C^2(\bar{\Omega})$ we have*

$$\int_\Omega v\triangle u \, \mathrm{d}y + \int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}y = \int_{\partial\Omega} v\nabla u \cdot N \, \mathrm{d}\sigma.$$

If we subtract the formula for interchanged u and v, then we obtain:

Green's Second Formula 3.16. *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for two functions $u, v \in C^2(\bar{\Omega})$ we have*

$$\int_\Omega v\triangle u - u\triangle v \, \mathrm{d}y = \int_{\partial\Omega} [v\nabla u - u\nabla v] \cdot N \, \mathrm{d}\sigma.$$

The significance of these formulas becomes apparent when we apply them to the fundamental solution $v(y) = \Phi(x-y)$. This function is harmonic for $y \neq x$, so we need to exclude a small ball $B(x,\epsilon)$. We apply Green's second formula on the domain $\Omega \setminus B(x,\epsilon)$. The left hand side becomes

$$\int_{\Omega\setminus B(x,\epsilon)} \Phi(x-y)\triangle u(y)\,\mathrm{d}y.$$

As argued in Theorem 3.2 (the part with I_ϵ) this integral is well defined in the limit $\epsilon \downarrow 0$. For the right hand side of Green's second formula, there are two boundary components to consider, namely $\partial\Omega$ and $\partial B(x,\epsilon)$. The integrals over $\partial B(x,\epsilon)$ are of a type L_ϵ and K_ϵ respectively. We have in the limit $\epsilon \downarrow 0$

$$\int_{\partial B(x,\epsilon)} \Phi(x-z)\nabla u(z)\cdot N(z)\,\mathrm{d}\sigma(z) \to 0.$$

For the other integral, we must be very careful of signs. As required by the divergence theorem, let N be the unit normal vector to $\partial B(x,\epsilon)$ that points towards x. It can be expressed as $N(z) = \frac{x-z}{|x-z|}$. Therefore $N(x-z') = \frac{z'}{|z'|}$ is the unit normal vector to $\partial B(0,\epsilon)$ pointing away from the origin. This is the opposite sign as the N in Theorem 3.2. We have

$$\begin{aligned}-\int_{\partial B(x,\epsilon)} u(z)\nabla_z(\Phi(x-z))\cdot N(z)\,\mathrm{d}\sigma(z) &= \int_{\partial B(x,\epsilon)} u(z)\nabla\Phi(x-z)\cdot N(z)\,\mathrm{d}\sigma(z)\\ &= \int_{\partial B(0,\epsilon)} u(x-z')\nabla\Phi(z')\cdot N(x-z')\,\mathrm{d}\sigma(z') \to -u(x).\end{aligned}$$

Rearranging the terms gives

Green's Representation Theorem 3.17. *Let the Divergence Theorem hold on the open and bounded domain $\Omega \subset \mathbb{R}^n$. Then for $x \in \Omega$ and a function $u \in C^2(\bar{\Omega})$ we have*

$$u(x) = -\int_\Omega \Phi(x-y)\triangle u(y)\,\mathrm{d}y + \int_{\partial\Omega} [\Phi(x-z)\nabla u(z) - u(z)\nabla_z(\Phi(x-z))]\cdot N\,\mathrm{d}\sigma(z).$$

This representation formula allows us to reconstruct a function u from its Laplacian and the values of u and the normal derivative $\nabla u \cdot N$ on $\partial\Omega$. But the Weak Maximum Principle implies the function is already uniquely determined by its Laplacian and boundary values, the normal derivatives on the boundary are redundant information. The question is, how can we calculate the normal derivatives from the other two pieces of information? If the domain Ω admits a function of the following type, then there is a clean formula.

Green's Function 3.18. *A function $G_\Omega : \{(x,y) \in \Omega\times\Omega \mid x \neq y\} \to \mathbb{R}$ is called Green's function for the open domain $\Omega \subset \mathbb{R}^n$, if it has the following two properties:*

(i) *For $x \in \Omega$ the function $y \mapsto G_\Omega(x,y) - \Phi(x-y)$ extends to a harmonic function on $y \in \Omega$.*

(ii) *For $x \in \Omega$ the function $y \mapsto G_\Omega(x,y)$ extends continuously to $\partial\Omega$ and vanishes on $y \in \partial\Omega$.*

It is worth considering how this definition applies to the special case $\Omega = \mathbb{R}^n$. Then the second condition is trivial and the first condition says that $G_{\mathbb{R}^n}(x,y) = \Phi(x-y) + u_x(y)$ is a Green's function for any family of harmonic functions $(u_x)_{x\in\Omega}$. In general, the first condition can also be expressed as $\triangle_y G_\Omega(x,y) = \delta_x$ in the sense of distributions, where δ_x is the delta distribution centered at $x \in \Omega$. This is equivalent because all harmonic distributions are harmonic functions.

Let's put them to use. We apply Green's Second Formula to the function $v(y) = G_\Omega(x,y) - \Phi(x-y)$. It is a harmonic function on all of Ω so there is no need to exclude a ball this time. Further, because we know the integrals with Φ are well defined, so therefore are the ones with G_Ω. We have

$$\int_\Omega G_\Omega(x,y)\triangle u(y)\,\mathrm{d}y - \int_\Omega \Phi(x-y)\triangle u(y)\,\mathrm{d}y$$
$$= -\int_{\partial\Omega} u(z)\nabla_z G_\Omega(x,z)\cdot N\,\mathrm{d}\sigma(z) - \int_{\partial\Omega} [\Phi(x-z)\nabla u(z) - u(z)\nabla_z(\Phi(x-z))]\cdot N\,\mathrm{d}\sigma(z).$$

Now Green's Representation Theorem implies

$$u(x) = -\int_\Omega G_\Omega(x,y)\triangle_y u(y)\,\mathrm{d}y - \int_{\partial\Omega} u(z)\nabla_z G_\Omega(x,z)\cdot N\,\mathrm{d}\sigma(z).$$

We should think of this as an improved version of Green's representation formula, enabled by the existence of a Green's function. We will shortly prove that conversely that if functions $f : \bar{\Omega} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ have sufficient regularity, then

$$u(x) := \int_\Omega G_\Omega(x,y)f(y)\,\mathrm{d}^{\mathrm{n}}y - \int_{\partial\Omega} g(z)\nabla_z G_\Omega(x,z)\cdot N\,\mathrm{d}\sigma(z)$$

defines a function that solves the Dirichlet Problem. Therefore the Dirichlet Problem reduces to the search of the Green's Function.

For bounded domains Ω there is a little more we can say about a Green's function. Firstly it is unique. If there are two Green's functions on Ω, then their difference is harmonic for all $y \in \Omega$:

$$G(x,y) - \tilde{G}(x,y) = G(x,y) - \Phi(x-y) - [\tilde{G}(x,y) - \Phi(x-y)]$$

and vanishes for $y \in \Omega$. By the weak maximum principle, this difference must be zero. As mentioned in the example above, there is no reason it needs to be unique on an unbounded domain. Further

Theorem 3.19 (Symmetry of the Green's Function)**.** *If there is a Green's Function G_Ω for the bounded domain Ω, then $G_\Omega(x,y) = G_\Omega(y,x)$ holds for all $x \neq y \in \Omega$.*

Proof. For $x \neq y \in \Omega$ let $\epsilon > 0$ be sufficiently small, such that both balls $B(x, \epsilon)$ and $B(y, \epsilon)$ are disjoint subsets of Ω. Green's Second Formula implies for the domain $\Omega \setminus (B(x, \epsilon) \cup B(y, \epsilon))$ and the functions $u(z) = G_\Omega(x, z)$ and $v(z) = G_\Omega(y, z)$

$$\begin{aligned}\int_{\partial B(x,\epsilon)} &[G_\Omega(y, z)\nabla_z G_\Omega(x, z) - G_\Omega(x, z)\nabla_z G_\Omega(y, z)] \cdot N \,\mathrm{d}\sigma(z) \\ &= \int_{\partial B(y,\epsilon)} [G_\Omega(x, z)\nabla_z G_\Omega(y, z) - G_\Omega(y, z)\nabla_z G_\Omega(x, z)] \cdot N \,\mathrm{d}\sigma(z).\end{aligned}$$

For $\epsilon \to 0$ the estimate for L_ϵ in the proof of Theorem 3.2 shows that both second terms converge to zero. The calculation of K_ϵ in the proof of Theorem 3.2 carries over and shows that the first terms converge to $G_\Omega(y, x)$ and $G_\Omega(x, y)$, respectively. □

Finding a Green's function for an arbitrary domain can be difficult, and they do not even exist for all domains. However it is feasible for highly symmetric domains, and the advantage is that then the solution has a concrete formula. We shall calculate Green's function for all balls in $\mathbb{R}^n$. Let us first restrict to the unit ball $\Omega = B(0, 1)$. The key is to try and add a harmonic function to $\Phi(x - y)$ that equals it on the boundary. We may use the inversion $x \mapsto \iota(x) = \frac{x}{|x|^2}$ in the unit sphere $\partial B(0, 1)$. It maps the inside of the unit ball to the outside and vice versa, fixing the boundary.

Green's Function of the unit ball 3.20. *The Green's Function of* $B(0, 1)$ *is*

$$G_{B(0,1)}(x, y) = \Phi(x-y)-\Phi(|x|(\iota(x)-y)) = \begin{cases} \Phi(x - y) - \Phi(\iota(x) - y) - \Phi(x) & \text{for } n = 2, \\ \Phi(x - y) - |x|^{2-n}\Phi(\iota(x) - y) & \text{for } n > 2. \end{cases}$$

Proof. Fix $x \in B(0, 1)$. There are two properties that we must satisfy. First the function $y \mapsto G_{B(0,1)}(x, y) - \Phi(x - y) = \Phi(|x|(\iota(x) - y))$ should extend to a harmonic function on all $y \in B(0, 1)$. Observe that $\iota(x)$ is a point outside unit ball, so $\iota(x) - y$ is never zero and thus this function is well-defined for all $y \in B(0, 1)$. Moreover, we have proved in a exercise that composing a harmonic function with rescaling, reflection or translation of its domain creates another harmonic function.

For the vanishing on the boundary, note that there is no problem extending $G_{B(0,1)}(x, y)$ for $y \in \partial B(0, 1)$, because x and $\iota(x)$ are not in $\partial B(0, 1)$. To show that it's zero, we need some geometry. For $|y| = 1$ we have

$$\begin{aligned}\big||x|(\iota(x) - y)\big|^2 &= (|x|^{-1}x - |x|y) \cdot (|x|^{-1}x - |x|y) = 1 - 2x \cdot y + |x|^2|y|^2 \\ &= |y|^2 - 2x \cdot y + |x|^2 = |x - y|^2.\end{aligned}$$

Because Φ is a function that only depends on the length of its argument, $\Phi(|x|(\iota(x) - y))$ and $\Phi(x - y)$ are equal on the boundary $y \in \partial B(0, 1)$. □

Although the definition of $G_{B(0,1)}$ appears to treat x and y differently, in fact $\big||x|(\iota(x)-y)\big|^2 = 1-2x\cdot y+|x|^2|y|^2$ from the above proof, which does not use on $|y|=1$, shows that the it is symmetric as expected.

The affine map $x\mapsto a+rx$ is a diffeomorphism from $B(0,1)$ onto $B(a,r)$ and a homeomorphism from $\partial B(0,1)$ onto $\partial B(a,r)$. We can use this coordinate change to transform a Dirichlet problem on the ball $B(a,r)$ to one on $B(0,1)$. If u solves $-\triangle u = f$ on $B(a,r)$ and $u|_{\partial B(a,r)} = g$ then $v(x) = u(a+rx)$ solves $-\triangle v = r^2 f(a+rx)$ on $B(0,1)$ and $v(x) = g(a+rx)$ for $x\in\partial B(0,1)$. The same is true in reverse. Thus the ability to solve the Dirichlet on one ball confers the ability to solve the Dirichlet problem on every ball (and the same for other domains related by similarity).

We can use this insight to give the Green's function for a general ball. We use an equivalent characterisation of the Green's function: for every $x\in\Omega$ the harmonic difference $u(y) := G_\Omega(x,y)-\Phi(x-y)$ is a solution to the Dirichlet problem

$$\triangle u = 0 \text{ on } \Omega, \qquad u(y) = 0-\Phi(x-y) \text{ for } y\in\partial\Omega.$$

(This gives an alternative proof of uniqueness.) For $\Omega = B(a,r)$ and a point $x' = a+rx\in B(a,r)$ the related Dirichlet problem on the unit ball is $v(x) = u(a+rx)$ with $\triangle v = 0$ on $B(0,1)$ and

$$v(y) = -\Phi(x'-(a+ry)) = -\Phi(r(x-y)) = \begin{cases} -\Phi(x-y)-\frac{r}{2\pi} & \text{for } n=2\\ -r^{2-n}\Phi(x-y) & \text{for } n\geq 3\end{cases}$$

for $y\in\partial\Omega$. By linearity and since constant functions are harmonic, we can write down the unique solution on $B(0,1)$ by inspection:

$$v(y) = \begin{cases} -\Phi(|x|(\iota(x)-y))-\frac{r}{2\pi} & \text{for } n=2\\ -r^{2-n}\Phi(|x|(\iota(x)-y)) & \text{for } n\geq 3.\end{cases}$$

Putting this all together gives

$$\begin{aligned} G_{B(a,r)}(x',y') &= \Phi(x'-y')+u(y') = \Phi(r(x-y))+v(y)\\ &= \begin{cases} \Phi(x-y)+\frac{r}{2\pi}-\Phi(|x|(\iota(x)-y))-\frac{r}{2\pi} & \text{for } n=2\\ r^{2-n}\Phi(x-y)-r^{2-n}\Phi(|x|(\iota(x)-y)) & \text{for } n\geq 3\end{cases}\\ &= r^{2-n}\left[\Phi(x-y)-r^{2-n}\Phi(|x|(\iota(x)-y))\right]\\ &= r^{2-n}G_{B(0,1)}(\tfrac{x'-a}{r},\tfrac{y'-a}{r}).\end{aligned}$$

It remains to prove therefore that taking the Green's representation formula and inserting f and g with sufficient regularity does indeed define a solution to the Dirichlet problem. We do this only for the specific example of the unit ball, but by the above discussion an analogous result will hold for any ball.

Poisson's Representation Formula 3.21. *For $\Omega = B(0,1)$, $f \in C^2(\overline{\Omega})$ and $g \in C(\partial\Omega)$ the unique solution of the Dirichlet Problem on Ω is given by*

$$u(x) = \int_{B(0,1)} G_{B(0,1)}(x,y) f(y) \,\mathrm{d}^{\mathrm{n}}y - \int_{\partial B(0,1)} g(y) \nabla_y G_{B(0,1)}(x,y) \cdot y \,\mathrm{d}\sigma(y).$$

Proof. It suffices to consider the two cases $g = 0$ and $f = 0$ separately.

Consider $g = 0$ first. The essential point is the symmetry of the Green's function, so whatever properties hold in the second variable also hold in the first. From Theorem 3.2 we have function $v(x)$ that satisfies $-\triangle v = f$. Their difference has the formula

$$u(x) - v(x) = \int_{B(0,1)} \Big[G_{B(0,1)}(x,y) - \Phi(x-y) \Big] f(y) \,\mathrm{d}^{\mathrm{n}}y.$$

But the bracketed expression is harmonic in x and therefore $u - v$ is harmonic. This shows that $-\triangle u = -\triangle v = f$. Moreover, we know that $G_{B(0,1)}(x,y)$ is zero for $x \in \partial B(0,1)$ and hence so too is $u(x)$.

The $f = 0$ case is the new part. We define the Poisson kernel $K(x,y) := -\nabla_y G_{B(0,1)}(x,y) \cdot y$. By the Symmetry of the Green's Function the function $x \mapsto K(x,y)$ is harmonic. Hence for $f = 0$ the given function u is harmonic. It remains to show

$$u(x) = \int_{\partial B(0,1)} g(y) K(x,y) \,\mathrm{d}\sigma(y)$$

extends continuously to $x \in \partial B(0,1)$ and coincides there with $g(x)$. The issue is that the integral is over $y \in \partial B(0,1)$ so their is a singularity in the integration in this limit. We compute for $|y| = 1$ and $n > 2$ (the reader should check this same formula holds for $n = 2$ too):

$$\begin{aligned} K(x,y) &= \frac{-1}{n(n-2)\omega_n} y \cdot \nabla_y \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x|^{n-2} \left|\iota(x) - y\right|^{n-2}} \right) \\ &= \frac{1}{n\omega_n} y \cdot \left(\frac{y-x}{|x-y|^n} - \frac{|x|^2 (y - \iota(x))}{|x|^n \left|\iota(x) - y\right|^n} \right) \\ &= \frac{1 - x\cdot y - |x|^2 + x \cdot y}{n\omega_n |x-y|^n} = \frac{1 - |x|^2}{n\omega_n |x-y|^n}. \end{aligned}$$

This clearly shows the singularity at $y = x$ but that for all other $x \in \partial B(0,1)$ it is zero. We observe

(i) the integral kernel $K(x,y)$ is positive for $(x,y) \in B(0,1) \times \partial B(0,1)$.

(ii) The following formula, which follows from Green's Representation Formula for the function $u = 1$ on the domain $\Omega = B(0,1)$:

$$\int_{\partial B(0,1)} K(x,y) \,\mathrm{d}\sigma(y) = 1 \quad \text{for} \quad x \in B(0,1).$$

(iii) For all $x \in \partial B(0,1)$, $\delta > 0$, and $y \in \partial B(0,1) \setminus B(x,\delta)$ there is the bound $K(\lambda x, y) \leq \frac{1}{n\omega_n \delta^n}(1-\lambda^2)$ Therefore the family of functions $y \mapsto K(\lambda x, y)$ converge uniformly to zero for $\lambda \uparrow 1$ on $y \in \partial B(0,1) \setminus B(x,\delta)$.

We will now prove that for continuous g the properties (i)-(iii) ensure that in the limit $\lambda \uparrow 1$ the family of functions $x \mapsto \int_{\partial B(0,1)} g(y) K(\lambda x, y)\, \mathrm{d}\sigma(y)$ converge on $\partial B(0,1)$ uniformly to g. For any $x \in \partial B(0,1)$, $0 < \lambda < 1$, and $\delta > 0$ we have estimate

$$\begin{aligned}
|u(\lambda x) - g(x)| &= \left| \int_{\partial B(0,1)} g(y) K(\lambda x, y) - g(x) K(\lambda x, y)\, \mathrm{d}\sigma(y) \right| && \text{using (ii)} \\
&\leq \int_{\partial B(0,1)} |g(y) - g(x)| K(\lambda x, y)\, \mathrm{d}\sigma(y) && \text{using (i)} \\
&= \left(\int_{\partial B(0,1) \setminus B(x,\delta)} + \int_{\partial B(0,1) \cap B(x,\delta)} \right) |g(y) - g(x)| K(\lambda x, y)\, \mathrm{d}\sigma(y) \\
&\leq \sup_{y \in \partial B(0,1)} |g(y) - g(x)| \times (1-\lambda^2)\delta^{-n} && \text{using (iii)} \\
&\quad + \sup_{y \in \partial B(0,1) \cap B(x,\delta)} |g(y) - g(x)| \times 1 && \text{using (ii).}
\end{aligned}$$

Therefore for any $\delta > 0$ and $0 < \lambda < 1$ we have the uniform estimate

$$\|u(\lambda x) - g(x)\|_\infty \leq (1-\lambda^2)\delta^{-n} \sup_{x,y \in \partial B(0,1)} |g(y) - g(x)| + \sup_{\substack{x \in \partial B(0,1) \\ y \in \partial B(0,1) \cap B(x,\delta)}} |g(y) - g(x)|.$$

Taking the limit $\lambda \uparrow 1$ we see that the limit is bounded by the second term for any $\delta > 0$, since the first term tends to zero. But the second term can be arbitrarily small, and therefore the uniform limit must be zero. This proves the claim. □

A harmonic function u on $B(a,r)$ which extends continuously to $\partial B(a,r)$ obeys

$$u(x) = \frac{r^2 - |x-a|^2}{nr\omega_n} \int_{\partial B(a,r)} \frac{u(y)}{|x-y|^n}\, \mathrm{d}\sigma(y).$$

Like the Weak Maximum Principle, this shows that u is completely determined by the values on $\partial B(a,r)$, except here the result is constructive. One can also integrate this formula in x over a ball, and after interchanging the integral and using some geometry, arrive at the Mean Value property.

One new consequence of this formula is an additional regularity result for harmonic functions. The dependence on x in the formula is well-behaved for $x \in B(a,r')$ with $r' < r$, because $|x-y|^{-n}$ is bounded away from its singularity. Therefore partial derivatives of u with respect to x can be expressed with similar formulas depending only on the values of u on a fixed ball $B(a,r)$. For all $y \in \partial B(a,r')$ the Taylor series of $x \mapsto |x-y|^{-n} = (y^2 - 2xy + x^2)^{-\frac{n}{2}}$ in $x = z$ converges uniformly to $|x-y|^{-n}$. This implies:

Corollary 3.22. *Harmonic functions on an open domain $\Omega \subset \mathbb{R}^n$ are analytic.* □

Another regularity result, which speaks to the connection between harmonic functions and holomorphic functions (if you know some complex analysis), is the so called 'removable singularities' theorem:

Lemma 3.23. *Let $\Omega \subset \mathbb{R}^n$ be an open neighbourhood of 0 and u a bounded harmonic function on $\Omega \setminus \{0\}$. Then u extends as a harmonic function to Ω.*

Proof. On a ball $B(0,r)$ with compact closure in Ω, Theorem 3.21 gives a harmonic function $\tilde{u}$ which coincides on $\partial B(0,r)$ with u. The family of harmonic functions $u_\epsilon(x) = \tilde{u}(x) - u(x) + \epsilon G_{B(0,r)}(x,0)$ on $B(0,r) \setminus \{0\}$ vanish on $\partial B(0,r)$. If for any $\epsilon > 0$ the function u_ϵ takes on $B(0,r) \setminus \{0\}$ a negative value, then due to the boundedness of u and $\tilde{u}$ and the unboundedness of $G_{B(0,r)}(\cdot,0)$ the harmonic function u_ϵ has a negative minimum on $B(0,r) \setminus \{0\}$. This contradicts the Strong Maximum Principle. Hence u_ϵ is non-negative. Analogously u_ϵ us for negative ϵ non-positive. Otherwise u_ϵ would have a positive maximum in $B(0,r) \setminus \{0\}$. In both limits $\epsilon \downarrow 0$ and $\epsilon \uparrow 0$ $u_0 = \tilde{u} - u$ vanishes identically on $B(0,r) \setminus \{0\}$ and $\tilde{u}$ is a harmonic extension of u to Ω. □

The proof shows a slightly stronger statement. Each harmonic function on $\Omega \setminus \{0\}$ whose absolute value $|u(x)|$ is for all $\epsilon > 0$ bounded by $\epsilon G_{B(0,r)}(x,0)$ on $B(0,\delta) \setminus \{0\}$ with sufficiently small $\delta > 0$ depending on ϵ has an harmonic extension to Ω.

Chapter 4

Heat Equation

In this chapter we investigate the heat equation

$$\dot{u} - \triangle u = 0$$

and the corresponding inhomogeneous variant

$$\dot{u} - \triangle u = f.$$

The unknown function u is defined on an open domain $\Omega \times (0, T) \subset \mathbb{R}^n \times \mathbb{R}$. We shall extend some statements about harmonic functions to solutions of the heat equation, but also try to understand the important differences. There is no widely agree upon name for solutions to the homogeneous heat equation, similar to harmonic functions for the Laplace equation, though some books use the term caloric. A previous class suggested to call them flames, similar to how solutions of the wave equations are waves, which I find cute.

There are two boundary value problems that we will examine in particular. The first is the initial value problem on $\mathbb{R}^n \times (0, T)$

$$\dot{u} - \triangle u = f \text{ on } \mathbb{R}^n \times (0, T), \qquad u(x, 0) = h(x) \text{ on } \mathbb{R}^n.$$

This is sometimes called the Cauchy problem. It purports to model how the temperature within an infinitely large body changes given the initial temperature h at every point. The inhomogeneous term f represents the infusion or removal of heat at points within the body. The second problem applies to a bounded spatial domain Ω

$$\dot{u} - \triangle u = f \text{ on } \Omega \times (0, T), \qquad u = g \text{ on } \partial\Omega \times [0, T], \qquad u(x, 0) = h(x) \text{ on } \Omega.$$

This problem is called the Dirichlet problem, in analogy to the corresponding problem for the Laplace equation. This models the temperature within a finite body but where additionally the temperature of the boundary is also controlled (specified by g). In both problems any solution should at least extend continuously to the boundary, so that the boundary conditions are meaningful.

The heat equation describes a diffusion process. This means a time-like evolution of space-like distributed quantities like heat or chemical concentration, or even probability. Let us provide a short justification of the equation as a model of heat. We have seen for the mean value property that the Laplacian measures the difference of a function from its mean value: for small r from the proof of Theorem 3.5 we have $\mathcal{S}'(r) \approx n^{-1} r \triangle u(x)$ which implies $\mathcal{S}(r) - u(x) \approx \frac{1}{2n} r^2 \triangle u(x)$. If the temperature u at x is cooler than the points around it, then $\dot{u}$ should be positive, and vice-versa if u is hotter. Moreover we have seen from the general conservation law (with $F = \nabla u$) that the quantity u is preserved by the heat equation (under appropriate assumptions). The simplicity of the equation together with these properties make it a useful model to study.

Before we begin the develop the theory that we will use, let's study some monstrous examples, to show us what to be wary of. The first shows the importance of the negative sign in the heat equation. We give an illustration that the heat equation is not time symmetric in the way that many models in physics are (at least conceptually) and that the 'reverse time' problem is not well-posed. Consider $n = 1$ and for any integer m define the function

$$u_m(x,t) = e^{m^2(T-t)} \sin mx.$$

They have the property that $\dot{u}_m = -m^2 u_m$ as well as $\partial_x^2 u_m = -m^2 u$. Therefore they all solve the homogeneous heat equation with 'terminal' condition $u_m(T) = \sin mx$. This example can even be applied to a Dirichlet-type problem. Consider the spatial domain $\Omega = (0, 2\pi)$ with the boundary values $g \equiv 0$. Because m is an integer, all these functions satisfy it. Even though these boundary conditions are smooth and uniformly bounded by 1, the solutions at any time $t < T$ can still be arbitrarily large

$$\sup |u_m(\cdot, t)| = e^{m^2(T-t)}.$$

This is one reason to only study the forward time Dirichlet problem.

Similarly for the Cauchy problem introduced above, there is also the possibility of rapidly growing solutions. Again for $n = 1$, we make the ansatz

$$u(x,t) = \sum_{l=0}^{\infty} g_l(t) x^l, \qquad \dot{u}(x,t) - \triangle u(x,t) = \sum_{l=0}^{\infty} (\dot{g}_l(t) - (l+2)(l+1) g_{l+2}(t)) x^l.$$

Thus if u solves the heat equation then we must have a recursion relation between g_l and g_{l+2}. For a given function $g_0(t) = g(t)$ and setting $g_1(t) \equiv 0$ we thus obtain the following formal solution of the homogeneous heat equation:

$$u(x,t) = \sum_{l=0}^{\infty} \frac{g^{(l)}(t)}{(2l)!} x^{2l}.$$

We now show that for $g(t) = \exp(-t^{-2})$ this power series indeed converges to a smooth solution and further that on every compact subset of $\mathbb{R}^n$ the uniform limit of this solution vanishes as $t \downarrow 0$. We first calculate $g^{(l)}(t)$ for any $l \in \mathbb{N}_0$ by a real polynomial p_l of degree l solving the relation

$$g^{(l)}(t) = t^{-l} p_l(t^{-2}) \exp(-t^{-2}) \quad \text{with} \quad p_{l+1}(z) = 2zp_l(z) - lp_l(z) - 2zp_l'(z).$$

This recursion relation for p_l follows by differentiating by t. The first two polynomials are $p_0(z) = 1$ and $p_1(z) = 2z$. We claim that the coefficient of $p_l(z)$ in front of z^k is bounded by $\frac{l!7^l}{2^k k!}$. For $l = 0$, $k = 0$ this is clear. By induction we obtain with $k \le l+1$

$$2\frac{l!7^l}{2^{k-1}(k-1)!} + l\frac{l!7^l}{2^k k!} + 2k\frac{l!7^l}{2^k k!} = \frac{l!7^l(4k+l+2k)}{2^k k!} \le \frac{l!7^l 7(l+1)}{2^k k!} \le \frac{(l+1)!7^{l+1}}{2^k k!}.$$

This proves the claim. Using the inequalities $\frac{l!}{(2l)!} = \frac{1}{2^l 1\cdot 3 \cdots (2l-1)} \le \frac{1}{2^l l!}$ we conclude

$$|u(x,t)| \le \sum_{l=0}^{\infty} \frac{l!7^l x^{2l}}{(2l)!t^l} \sum_{k=0}^{l} \frac{g(t)}{2^k k! t^{2k}} \le \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{7x^2}{2t}\right)^l \sum_{k=0}^{\infty} \frac{g(t)}{k!}\left(\frac{1}{2t^2}\right)^k = \exp\left(\frac{7x^2}{2t} - \frac{1}{2t^2}\right).$$

Therefore the series converges absolutely and for $t \downarrow 0$ uniformly on compact sets to 0. This means that we can extend u smoothly to $t \le 0$ by giving it the value 0. This means that the Cauchy problem with initial value $h \equiv 0$ has a non-zero solution: The space is the same temperature everywhere and suddenly wild temperature fluctuations begin. This shows that even though it seems as if the Cauchy problem should be well-posed, additional constraints will be required.

4.1 Spectral Theory and the Fourier Transform

Let us give some motivation for introducing spectral theory: the theory of the eigenvalues of the operator $-\triangle$. Let us look for 'separable' solutions of the homogeneous heat equation, solutions that neatly factorise as $u(x,t) = \varphi(t)h(x)$. These solve the heat equation if

$$\dot{\varphi}(t)h(x) - \varphi(t)\triangle h(x) = 0 \qquad \Leftrightarrow \qquad \frac{\dot{\varphi}(t)}{\varphi(t)} = \frac{\triangle h(x)}{h(x)}.$$

Clearly the only way that this is possible is if the two sides are equal to some constant $-\lambda$. This means that h is an eigenfunction of the (negative) Laplace operator:

$$-\triangle h = \lambda h \quad \text{on} \quad \Omega,$$

and $\dot{\varphi} = -\lambda\varphi$. The factorisation is only determined up to a scaling, so we set $\varphi(0) = 1$. Thus $\varphi(t) = e^{-\lambda t}$ and u has the initial value $u(x,0) = h(x)$.

Turning this around, if we are given an initial value problem where h is an eigenfunction of the Laplacian, then this method gives a solution. More generally, if the initial condition is a linear combination of eigenfunctions then a linear combination of separable solutions solves the problem. The question now arises can every function be written as a linear combination of eigenfunctions in some suitable sense?

What are the eigenfunctions of $-\triangle$? The trigonometric functions provide many examples for every $\lambda > 0$:

$$-\triangle e^{2\pi i k\cdot x} = 4\pi^2|k|^2 e^{2\pi i k\cdot x}.$$

The drawback of these functions are that they are not integrable on the plane because they have modulus 1 at every point. But in a limiting sense they are all orthogonal to one another in L^2 inner product

$$\langle e^{2\pi i k_1\cdot x}, e^{2\pi i k_2\cdot x}\rangle = \int_{\mathbb{R}^n} e^{2\pi i k_1\cdot x}\overline{e^{2\pi i k_2\cdot x}}\,\mathrm{d}^{\mathrm{n}}x = \int_{\mathbb{R}^n} e^{2\pi i(k_1-k_2)\cdot x}\,\mathrm{d}^{\mathrm{n}}x = 0$$

because the integrand is periodic and the integral over a single period is zero. This leads us to define the Fourier transform as the coefficients of the orthogonal projection of a function onto these functions, in the sense that for a finite dimensional inner product space $h = \sum\langle h, e_i\rangle e_i$ for an orthonormal basis $\{e_i\}$.

Definition 4.1. *The Fourier transform of a function $h : \mathbb{R}^n \to \mathbb{R}$ is defined to be*

$$\hat{h}(k) = \mathscr{F}[h](k) := \int_{\mathbb{R}^n} e^{-2\pi i k\cdot x} h(x)\,\mathrm{d}^{\mathrm{n}}x.$$

Be aware: there are several definitions of Fourier transform that differ by a constant scaling and a scaling of k. Always check which is being used.

When one learns Fourier analysis in detail, a major theme is under what conditions this definition makes sense, how it can be extended to other classes of functions, and which of the important properties are retained for these extensions. For example, a basic result that we will soon prove is that if the function $h \in L^1(\mathbb{R}^n)$ then its Fourier transform is continuous and bounded.

Let us compute the Fourier transform for an important example: the Gaussian curve $e^{-|\pi x|^2}$. We begin

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-2\pi i k\cdot x} e^{-|\pi x|^2}\,\mathrm{d}^{\mathrm{n}}x &= \int_{\mathbb{R}^n} e^{-|k|^2+|k|^2-2ik\cdot(\pi x)-|\pi x|^2}\,\mathrm{d}^{\mathrm{n}}x = \int_{\mathbb{R}^n} e^{-|k|^2-(ik+\pi x)\cdot(ik+\pi x)}\,\mathrm{d}^{\mathrm{n}}x \\ &= e^{-|k|^2}\int_{\mathbb{R}^n} e^{-(ik+\pi x)\cdot(ik+\pi x)}\,\mathrm{d}^{\mathrm{n}}x = \pi^{-n}e^{-|k|^2}\int_{ik+\mathbb{R}^n} e^{-y\cdot y}\,\mathrm{d}^{\mathrm{n}}y.\end{aligned}$$

To finish we need to compute the value of the final integral. It is so famous that it has its own name 'the Gaussian integral'. It value is $\pi^{n/2}$. Several methods to compute this will be explored in the tutorial. By rescaling we also have the Fourier transforms for other Gaussians. In conclusion

$$\mathscr{F}[e^{-a|x|^2}](k) = \left(\frac{\pi}{a}\right)^{n/2} e^{-\frac{1}{a}|\pi k|^2}.$$

One obvious class of functions that can be Fourier transformed is the test functions because they have compact support. But this turns out to be a little too restrictive. Instead we consider functions that decay rapidly at infinity.

Definition 4.2. *The Schwartz space $\mathscr{S}$ contains all smooth complex valued functions f on $\mathbb{R}^n$ for which $\rho_{l,\alpha}(f) := \sup |x|^{2l} |\partial^\alpha f(x)|$ are finite for all $l \in \mathbb{N}$ and all $\alpha \in \mathbb{N}_0^n$.*

There are other equivalent definitions in the literature. A common alternative is to use $(1+|x|^2)^l$ instead of $|x|^{2l}$. One characterisation of $\mathscr{S}$ is that it is the largest subspace of integrable functions that is closed under differentiation and multiplication with polynomials. For following lemma however is perhaps the more important justification for considering this space.

Lemma 4.3. *The Fourier transformation maps $\mathscr{S}$ onto $\mathscr{S}$. For any function $h \in \mathscr{S}$ and $\hat{h} = \mathscr{F}[h]$ we have*

$$\mathscr{F}[\partial_j h](k) = 2\pi i k_j \hat{h}(k), \qquad \text{and} \qquad \mathscr{F}[-2\pi i x_j h](k) = \partial_j \hat{h}(k).$$

Proof. If we simply take the absolute value of the definition of the Fourier transform. This gives us $|\hat{h}(k)| \le \int_{\mathbb{R}^n} |h(y)| \, \mathrm{d}^n y = \|h\|_{L^1(\mathbb{R}^n)}$. Any $h \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$ certainly has finite L^1-norm and by taking supremum we obtain

$$\|\hat{h}\|_\infty \le \|h\|_{L^1(\mathbb{R}^n)}.$$

This shows that $\mathscr{F}$ is a continuous linear operator from $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with the L^1-norm to $C_b(\mathbb{R}^n, \mathbb{C})$ with the supremum norm. Since $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ is dense in $L^1(\mathbb{R}^n)$, the Fourier transform extends to a continuous linear map from $L^1(\mathbb{R}^n)$ into the Banach space $C_b(\mathbb{R}^n, \mathbb{C})$, as we claimed above.

But let us return to Schwarz functions and prove what is stated in the lemma. By integration by parts

$$\mathscr{F}[\partial_j h](k) = -\int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} \left(e^{-2\pi i k\cdot x}\right) h(x) \, \mathrm{d}^n x = -\int_{\mathbb{R}^n} (-2\pi i k_j) e^{-2\pi i k\cdot x} h(x) \, \mathrm{d}^n x = 2\pi i k_j \hat{h}(k).$$

To make this calculation rigorous, one should integrate by parts on a large cube $[-R, R]^n$. But the decay properties of h ensure that the boundary terms vanish in the limit. Applying this formula with higher derivatives gives a polynomial in k on the right. Turning this relation around proves that any polynomial times $\hat{h}$ is the Fourier transform of a Schwartz function and thus bounded.

Similarly we can differentiate $\hat{h}$:

$$\frac{\partial}{\partial k_j}\hat{h} = \int_{\mathbb{R}^n} \frac{\partial}{\partial k_j}\left(e^{-2\pi i k\cdot x}\right) h(x)\,\mathrm{d}^{\mathrm{n}}x = \int_{\mathbb{R}^n} e^{-2\pi i k\cdot x}(-2\pi i x_j h(x))\,\mathrm{d}^{\mathrm{n}}x = \mathcal{F}[-2\pi i x_j h(x)](k).$$

This is justified by the estimate

$$|\partial_j \hat{h}(k)| = \left|\int_{\mathbb{R}^n} -2\pi i x_j e^{-2\pi i k\cdot x} h(x)\,\mathrm{d}^{\mathrm{n}}x\right| \leq 2\pi \||x|h(x)\|_{L^1(\mathbb{R}^n)}.$$

Because h decays faster than any power of $|x|$ the right hand side is bounded. Repeatedly applying this differentiation formula shows that $\hat{h}$ is smooth. The combination of the differentiation and polynomial rules for the Fourier transform therefore proves that $\hat{h}$ is Schwartz. □

The property of transforming derivatives into polynomials is what makes the Fourier transform a useful tool in solving ODEs and PDEs. Let's see how it applies to the heat equation. The Fourier transform of the Laplacian is $\mathscr{F}[\triangle u] = (2\pi i)^2|k|^2\hat{h}$, where we only Fourier transform the space variables and leave t out from the integral. Under sufficient regularity assumptions a solution to the heat equation obeys

$$\mathscr{F}[\partial_t u] + 4\pi^2|k|^2\hat{u} = \partial_t\hat{u} + 4\pi^2|k|^2\hat{u} = 0$$

by interchanging the ∂_t and integration. For each value of k this is an ODE for $\hat{u}(k,t)$ in the variable t. We even get initial conditions by applying the Fourier transform to the initial condition of the PDE $\hat{u}(k,0) = \hat{h}(k)$. It has the solution

$$\hat{u}(k,t) = e^{-4\pi^2|k|^2 t}\hat{u}(k,0) = e^{-4\pi^2|k|^2 t}\hat{h}(k).$$

So if we are able to find a function that has this as its Fourier transform, we have solved the heat equation. For this we need to understand how the Fourier transform behaves with respect to products and convolutions.

Lemma 4.4. *Let $u, v \in \mathscr{S}$. Then $\mathscr{F}[u * v] = \hat{u}\hat{v}$ and $\mathscr{F}[uv] = \hat{u} * \hat{v}$.*

Proof. This follows by direct calculation.

$$\begin{aligned}
\mathscr{F}[u*v](k) &= \int_{\mathbb{R}^n} e^{-2\pi i k\cdot x}\left(\int_{\mathbb{R}^n} u(x-y)v(y)\,\mathrm{d}^{\mathrm{n}}y\right)\mathrm{d}^{\mathrm{n}}x \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-2\pi i k\cdot x}u(x-y)\,\mathrm{d}^{\mathrm{n}}x\right)v(y)\,\mathrm{d}^{\mathrm{n}}y \\
&= \int_{\mathbb{R}^n} e^{-2\pi i k\cdot y}\left(\int_{\mathbb{R}^n} e^{-2\pi i k\cdot z}u(z)\,\mathrm{d}^{\mathrm{n}}z\right)v(y)\,\mathrm{d}^{\mathrm{n}}y \\
&= \int_{\mathbb{R}^n} e^{-2\pi i k\cdot z}u(z)\,\mathrm{d}^{\mathrm{n}}z \int_{\mathbb{R}^n} e^{-2\pi i k\cdot y}v(y)\,\mathrm{d}^{\mathrm{n}}y = \hat{u}(k)\hat{v(k)}.
\end{aligned}$$

The second half of the lemma is an easy consequence of the first half together with the inverse Fourier transform, which is given after Theorem 4.7. We really only need the first half of the lemma, but it much prettier to present the two results side-by-side. □

Because of our earlier example, we know that

$$\mathscr{F}\left[\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}\right] = e^{-4\pi^2|k|^2 t}.$$

Therefore we can conclude that

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} *_x h(x)$$

is a solution to the heat equation with initial condition $u(x,0) = h(x)$, where the convolution is only taken over the spatial variables.

Our derivation of the solution has assumed that the functions in question have sufficient regularity such that we were able to interchange the order of integration or differentiate under the integral sign as needed. In the next section we will take the formula for the solution that we have derived and prove directly, under weaker assumptions on h, that it solves the Cauchy problem.

4.2 Fundamental Solution

Our method of the previous section to solve the homogeneous heat equation through a Fourier transform uncovered a particular Gaussian function. It turns out to be a fundamental solution for the heat equation that is well-suited to the case $t > 0$, which holds for both problems we are interested in.

Definition 4.5. *The fundamental solution of the heat equation is defined as*

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} & \text{for} \quad x \in \mathbb{R}^n, t > 0 \\ 0 & \text{for} \quad x \in \mathbb{R}^n, t \leq 0 \end{cases}.$$

For $t \neq 0$ one can check that this solves the homogeneous heat equation be direct calculation (Exercise). For $x \neq 0$ we also know that $t \mapsto \Phi(x,t)$ is a smooth function, so in fact Φ solves the heat equation in the strong sense everywhere except $(0,0)$. We will show that $(\partial_t - \triangle)\Phi = \delta$ soon. Similar to the fundamental solution of the Laplace equation, this fundamental solution has the scaling property $\Phi(ax, a^2t) = a^{-n}\Phi(x,t)$. You may be wondering if the odd scaling factor for Φ is meaningful. It is, as the following lemma shows.

Lemma 4.6. *For all $t > 0$ the fundamental solution satisfies* $\displaystyle\int_{\mathbb{R}^n} \Phi(x,t)\,\mathrm{d}^n x = 1.$

Proof. $$\frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\,\mathrm{d}^n x = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-x^2}\,\mathrm{d}^n x = \frac{1}{\pi^{n/2}} \left(\int_{\mathbb{R}} e^{-x^2} dx\right)^n = 1.$$ □

We can therefore understand the fundamental solution as being similar to a mollifier on $\mathbb{R}^n$. As $t \downarrow 0$ the function grows and concentrates near the origin. It is not a mollifier because it does not have compact support, but it does lie in $\mathscr{S}$ and we should expect that the convolution with Φ converges in the limit $t \downarrow 0$ to the identity. This is the content of the following theorem. This theorem also gives a solution to the Cauchy problem for the homogeneous heat equation under the assumption that the initial condition is continuous and bounded.

Theorem 4.7. *For $h \in C_b(\mathbb{R}^n, \mathbb{R})$ the following function u has the properties (i)-(iii):*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) h(y)\,\mathrm{d}^n y$$

(i) $u \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$

(ii) $\dot{u} - \triangle u = 0$ *on* $\mathbb{R}^n \times \mathbb{R}^+$

(iii) u *extends continuously to* $\mathbb{R}^n \times [0,\infty)$ *with* $\lim_{t\to 0} u(x,t) = h(x)$.

Proof. For $t > 0$ by the smoothness of Φ and the boundedness of h, the function is well-defined and we can pass derivatives into the integral. This should that u is smooth. Likewise (ii) follows, since Φ solves the heat equation on $\mathbb{R}^n \times \mathbb{R}^+$.

The harder argument is (iii). For any $\epsilon > 0$ and any x in a compact subset of $\mathbb{R}^n$ there exists $\delta > 0$, such that $|h(x) - h(y)| < \epsilon$ for all $|x - y| < \delta$ (continuity implies uniform continuity on any compact subset). Furthermore there exists $T > 0$, such that

$$\int_{\mathbb{R}^n \setminus B(0,\delta)} \Phi(y,t)\,\mathrm{d}^n y = \int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} \Phi(z,1)\,\mathrm{d}^n z < \epsilon \qquad \text{for all } 0 < t < T.$$

This implies

$$\begin{aligned} |u(x,t) - h(x)| &= \left| \int_{\mathbb{R}^n} \Phi(x-y,t)(h(y) - h(x))\,\mathrm{d}^n y \right| \\ &\le \int_{B(x,\delta)} \Phi(x-y,t) \mid h(y) - h(x) \mid \mathrm{d}^n y + \int_{\mathbb{R}^n \setminus B(x,\delta)} \Phi(x-y,t) |h(y) - h(x)|\,\mathrm{d}^n y \\ &\le \epsilon + 2\epsilon \sup\{|h(y)| \mid y \in \mathbb{R}^n\} \end{aligned}$$

for all $0 < t < T$. So $u(x,t)$ converges in the limit $t \downarrow 0$ uniformly on compact subsets of $\mathbb{R}^n$ to h. □

Part (iii) of this theorem is also an important lemma in Fourier analysis, because it leads to an explicit formula for the inverse of the Fourier transform. Suppose that $u, v \in \mathscr{S}$. We compute the following integral parameterised in x

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{u}(k)v(k)e^{2\pi i k\cdot x}\,\mathrm{d^n}k &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} u(y)e^{-2\pi i k\cdot(y-x)}\,\mathrm{d^n}y\right)v(k)\,\mathrm{d^n}k \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} u(z+x)e^{-2\pi i k\cdot z}\,\mathrm{d^n}z\right)v(k)\,\mathrm{d^n}k = \int_{\mathbb{R}^n} u(z+x)\left(\int_{\mathbb{R}^n} v(k)e^{-2\pi i k\cdot z}\,\mathrm{d^n}k\right)\mathrm{d^n}z \\ &= \int_{\mathbb{R}^n} u(z+x)\hat{v}(z)\,\mathrm{d^n}z.\end{aligned}$$

The trick is to now choose $\hat{v}$ to be the fundamental solution $\Phi(x,\epsilon)$. This gives

$$\int_{\mathbb{R}^n} \hat{u}(k)e^{-4\pi^2|k|^2\epsilon}e^{2\pi i k\cdot x}\,\mathrm{d^n}k = \int_{\mathbb{R}^n} u(z+x)\Phi(z,\epsilon)\,\mathrm{d^n}z = \int_{\mathbb{R}^n} u(y)\Phi(y-x,\epsilon)\,\mathrm{d^n}y.$$

Taking the limit as $\epsilon \downarrow 0$ and applying Theorem 4.7(iii) on the right hand side proves

$$\int_{\mathbb{R}^n} \hat{u}(k)e^{2\pi i k\cdot x}\,\mathrm{d^n}k = u(x).$$

To summarise, the inverse Fourier transform is

$$\mathscr{F}^{-1}[u](x) = \int_{\mathbb{R}^n} u(k)e^{2\pi i k\cdot x}\,\mathrm{d^n}k = \mathscr{F}[u](-x).$$

The fact that the Fourier transform and its inverse differ only by a sign in the exponent of the exponential is the reason that it has so many 'dual' properties, such as for multiplication and convolution, or for differentiation and multiplication by polynomials.

The equation above for u and v is also the important step to extend the Fourier transform to (some) distributions. When $x = 0$ we have

$$\int_{\mathbb{R}^n} \hat{u}(k)v(k)\,\mathrm{d^n}k = \int_{\mathbb{R}^n} u(z)\hat{v}(z)\,\mathrm{d^n}z.$$

If this was written in the notation of distributions it would be $F_{\hat{u}}(v) = F_u(\hat{v})$. This seems as if it would be a suitable definition of the Fourier transform of a distribution. However, even if v is a test function, we can't be sure that $\hat{v}$ is a test function only that it is Schwartz, and thus $F(\hat{v})$ is not defined for all distributions.

Unfortunately there is no way to fix this. Instead we must restrict ourselves to consider only distributions that can act on Schwartz functions. But what does this mean? First we recognise that $\sup \rho_{l,\alpha}$ from Definition 4.2 of $\mathscr{S}$ constitutes a family of seminorms for Schwartz space. Further the inclusion of the space of test functions $\mathcal{D}$ into the Schwartz space $\mathscr{S}$ is continuous and dense with respect to this topology. Therefore we can identify the subspace of distributions that can be extended continuously to act on $\mathscr{S}$.

Definition 4.8. *Let $F \in \mathcal{D}'$ be a distribution. Suppose that ϕ_m is a sequence of test functions that converges to zero in $\mathscr{S}$, i.e. $\lim_{m\to\infty} \rho_{l,\alpha}(\phi_m) = 0$ for all l, α. We say that F is a tempered distribution $F \in \mathscr{S}'$ if $\lim_{m\to\infty} F(\phi_m) = 0$. If F is a tempered distribution then it acts on a Schwartz function ϕ by*

$$F(\phi) = \lim_{m\to\infty} F(\phi_m)$$

for any sequence of test functions ϕ_m that converges to ϕ in $\mathscr{S}$. For tempered distributions, we define the Fourier transform $\hat{F}(\phi) = F(\hat{\phi})$.

Many of the properties of Fourier transforms on $\mathscr{S}$ carry over to $\mathscr{S}'$, in particular the differentiation and polynomial multiplication rules. Defining the Fourier transform on distributions is not just a convenient way to extend it to a large class of functions but actually essential for understanding the Fourier transforms of many common functions. For example, the Fourier transform of the constant function 1 is the delta distribution.

Fourier analysis can also solve the inhomogeneous heat equation on $\mathbb{R}^n \times \mathbb{R}^+$. Taking the transform of the PDE results in the inhomogeneous ODE

$$\partial_t \hat{u} + 4\pi^2 |k|^2 \hat{u} = \hat{f}.$$

This has the solution

$$\hat{u}(k,t) = e^{-4\pi^2|k|^2 t}\hat{h}(k) + \int_0^t e^{-4\pi^2|k|^2(t-s)} \hat{f}(k,s)\,\mathrm{d}s.$$

We recognise the first term from the homogeneous case. The second term is new, but it is the integral over time of the product of $\hat{\Phi}(k, t-s)$ and $\hat{f}$. Performing the inverse transform suggests the following solution

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)h(y)\,\mathrm{d^n}y + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)\,\mathrm{d^n}y\,\mathrm{d}s.$$

It remains to consider the regularity of the second integral.

Theorem 4.9 (Solution of the inhomogeneous heat equation)**.** *If f is twice continuously and bounded differentiable on $\mathbb{R}^n \times [0,\infty)$, then*

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)\,\mathrm{d^n}y\,\mathrm{d}s$$

solves the inhomogeneous initial value problem

$$\dot{u} - \triangle u = f \text{ on } \mathbb{R}^n \times \mathbb{R}^+ \quad \text{and} \quad \lim_{t\to 0} u(x,t) = 0.$$

Proof. The integrand has a singularity when $s = t$. Therefore consider

$$u_\epsilon(x,t) = \int_0^{t-\epsilon} \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, \mathrm{d}^{\mathrm{n}} y\, \mathrm{d}s$$

To this function we can apply the heat equation with impunity:

$$\begin{aligned}
&\dot{u}_\epsilon(x,t) - \triangle u_\epsilon(x,t) \\
&= \int_{\mathbb{R}^n} \Phi(x-y, t-(t-\epsilon)) f(y, t-\epsilon)\, \mathrm{d}^{\mathrm{n}} y + \int_0^{t-\epsilon} \int_{\mathbb{R}^n} (\partial_t - \triangle) \Phi(x-y, t-s) f(y,s)\, \mathrm{d}^{\mathrm{n}} y\, \mathrm{d}s \\
&= \int_{\mathbb{R}^n} \Phi(x-y, \epsilon) f(y, t-\epsilon)\, \mathrm{d}^{\mathrm{n}} y.
\end{aligned}$$

Theorem 4.7 (iii) implies $\lim_{\epsilon \to 0} \dot{u}_\epsilon - \triangle u_\epsilon = f$ on $\mathbb{R}^n \times \mathbb{R}^+$. Additionally $u_\epsilon(x, \epsilon) = 0$. The assumptions on f are sufficient to conclude that

$$f = \lim_{\epsilon \to 0} (\dot{u}_\epsilon(x,t) - \triangle u_\epsilon(x,t)) = \left(\frac{\partial}{\partial t} - \triangle \right) \lim_{\epsilon \to 0} u_\epsilon(x,t) = \left(\frac{\partial}{\partial t} - \triangle \right) u(x,t)$$

and $0 = \lim_{\epsilon \to 0} u_\epsilon(x, \epsilon) = u(x, 0)$. Properly one should bound the difference between u and u_ϵ, which is the integral in time over the short interval $[t - \epsilon, t]$, in a similar manner to Theorem 3.2. □

We summarise our inquiries with the following statement.

Corollary 4.10. *Suppose f is twice continuously and bounded differentiable on $\mathbb{R}^n \times [0, \infty)$ and h is continuous and bounded on $\mathbb{R}^n$. The inhomogeneous initial value problem has the following solution:*

$$\begin{aligned}
\dot{u} - \triangle u &= f & u(x,0) &= h(x) \\
u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y, t) h(y)\, \mathrm{d}^{\mathrm{n}} y & + \int_0^t \int_{\mathbb{R}^n} &\Phi(x-y, t-s) f(y,s)\, \mathrm{d}^{\mathrm{n}} y\, \mathrm{d}s.
\end{aligned}$$

□

To finish the section we make some qualitative remarks on the behaviour of these solutions. The two integrals are a homogeneous solution that satisfies the initial condition 0 and an inhomogeneous solution that vanishes initially. One is reminded of the Green's representation formula, which was also two integrals dividing the task between themselves. We can also see that as a physics model of heat it violates the principle of locality and the speed of light. Consider $f = 0$, so there is no additional sources of heat, and suppose the initial temperature h is non-negative and has compact support. Then for any point and time $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ the solution is positive, because Φ is everywhere positive. The interpretation is that the heat that was present in the support of h has instantly spread out to the whole space.

4.3 Maximum Principle

Like elliptic PDEs, parabolic PDEs also have a maximum principle. In this section we will prove a weak maximum principle for the heat equation and apply it to the question of uniqueness of the Dirichlet and Cauchy problems. There is an approach to the maximum principle based on so-called 'heat balls' that mimic the mean value property for the Laplace equation (see Evans), but this is computationally messy. Instead we follow Han and give a proof in the style of Theorem 3.13.

The domain of the heat equation distinguishes time and spatial directions. We therefore make special definitions adapted to this distinction. For any open domain $\Omega \subset \mathbb{R}^n$ we define the parabolic cylinder as $\Omega_T = \Omega \times (0, T]$. The parabolic boundary $\partial\Omega_T$ of Ω_T is defined as $\bar{\Omega}_T \setminus \Omega_T$. It is the union of $(\partial\Omega \times (0, T]) \cup \big(\bar{\Omega} \times 0\big)$ and does not contain at time $t = T$ points inside of Ω.

Theorem 4.11 (Weak maximum principle for the heat equation)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and u a twice differentiable function on Ω_T that extends continuously to $\bar{\Omega}_T$. Suppose that u is a subsolution to the heat equation:*

$$\dot{u} - \triangle u \leq 0$$

on Ω_T. Then the maximum of u is taken on $\partial\Omega_T$.

Proof. Note because Ω is bounded that $\bar{\Omega}_T$ is compact, and thus u must have a maximum. The theorem claims that the maximum occurs on the boundary, but does not forbid it from also occurring on the interior. The constant function would be an example where the maximum is taken both on the boundary and the interior.

We first prove the theorem under the stronger assumption that $\dot{u} - \triangle u < 0$. Suppose that u has a maximum at $(x_0, t_0) \in \Omega_T$. If $t_0 < T$ then we can also say that $\partial_t u(x_0, t_0) = 0$, otherwise if $t = T$ we can only say that $\partial_t u(x_0, t_0) \geq 0$. In either case we see that $0 > \dot{u}(x_0, t_0) - \triangle u(x_0, t_0) \geq -\triangle u(x_0, t_0)$. Also because this point is a maximum $\nabla_x u(x_0, t_0) = 0$ and the Hessian H in the spatial coordinates is negative semidefinite. As argued in Theorem 3.13 at such a point $\triangle u(x_0, t_0) \leq 0$. But now we have a contradiction. Therefore the maximum cannot occur on Ω_T.

Next we handle the general case with a trick similar to Theorem 3.13. For any $\epsilon > 0$ define

$$u_\epsilon(x, t) := u(x, t) - \epsilon t.$$

This forces

$$(\partial_t - \triangle)u_\epsilon = \dot{u} - \triangle u - \epsilon \leq -\epsilon < 0.$$

Thus the special case applies to u_ϵ and we conclude that the maximum of u_ϵ occurs on the boundary. But we can now argue

$$\max_{\bar{\Omega}_T} u = \max_{\bar{\Omega}_T}(u_\epsilon + \epsilon t) \le \max_{\bar{\Omega}_T} u_\epsilon + \epsilon T = \max_{\partial\Omega_T} u_\epsilon + \epsilon T \le \max_{\partial\Omega_T} u + \epsilon T.$$

Taking $\epsilon \downarrow 0$ yields the result. □

The following is an easy consequence, similar to the uniqueness of the Dirichlet problem for the Laplace equation.

Theorem 4.12. *On an open and bounded domain $\Omega \subset \mathbb{R}^n$ there exists at most one solution u of the Dirichlet problem for the inhomogeneous heat equation.*

Proof. Suppose that there were two solutions. Consider their difference u. This function must solve the homogeneous heat equation and vanishes on both the initial boundary $\Omega \times \{0\}$ and the spatial boundary $\partial\Omega \times (0, T)$. In other words, it is zero on the parabolic boundary. By the weak maximum principle applied to u and $-u$ the maximum and minimum of u is zero. Thus $u \equiv 0$ and the two solutions are equal. □

We can also conclude the 'comparison principle' or 'monotonicity property' for the heat equation: If one body starts hotter than another at every point $h_1 \ge h_2$, stays hotter on the boundary $g_1 \ge g_2$ and receives more heat on the interior $f_1 \ge f_2$, then at every point and every time the first body is hotter than the second.

Remarkably we can also use the weak maximum principle to show a form of uniqueness in the Cauchy problem, even though it is on a unbounded domain. We must be careful however, as we have seen that the solution is not unique: we began the chapter with the example of a function that is identically zero initially and then springs to life. Any such example however must be a monster.

Theorem 4.13. *Let u be a solution on $\mathbb{R}^n \times (0, T]$ of the Cauchy problem:*

$$\dot{u} - \triangle u = 0 \ \text{on } \mathbb{R}^n \times (0, T) \qquad\qquad u(x, 0) = 0 \ \text{on } \mathbb{R}^n \times \{0\},$$

which is bounded by $|u(x,t)| \le Me^{A|x|^2}$ on $\mathbb{R}^n \times [0, T]$ for some positive constants $A, M > 0$. Then u is identically zero.

Proof. Choose $a > A$. We will prove that $u \equiv 0$ on $\mathbb{R}^n \times [0, \frac{1}{4a}]$. The result then holds on $[0, T]$ by induction on the decomposition $[0, T] = [0, \frac{1}{4a}] \cup [\frac{1}{4a}, \frac{2}{4a}] \cup \ldots$.

For any $R > 0$, define the function

$$v_R(x, t) = \frac{Me^{-(a-A)R^2}}{(1 - 4at)^{\frac{n}{2}}} \exp\left(\tfrac{a|x|^2}{1-4at}\right)$$

on $B(0,R)\times(0,\frac{1}{4a})$. It is an easy check that v_R solves the homogeneous heat equation and it is clearly positive. Moreover, on the sphere $x\in\partial B(0,R)$ it is larger than u, since

$$v_R = \frac{Me^{-(a-A)R^2}}{(1-4at)^{\frac{n}{2}}}\exp\left(\tfrac{aR^2}{1-4at}\right) \geq Me^{-(a-A)R^2}\exp\left(aR^2\right) = Me^{AR^2} \geq |u|$$

Hence by the maximum principle we know that $v_R \geq |u|$ on all of $\overline{B(0,R)}\times[0,\frac{1}{4a}]$.

Now choose any point $(x,t)\in\mathbb{R}^n\times[0,\frac{1}{4a}]$. For all $R>|x|$ we know that $|u(x,t)|<v_R(x,t)$. But

$$\lim_{R\to\infty} v_R(x,t) = \frac{M}{(1-4at)^{\frac{n}{2}}}\exp\left(\tfrac{a|x|^2}{1-4at}\right)\lim_{R\to\infty} e^{-(a-A)R^2} = 0.$$

Thus $u(x,t)=0$ too. □

The obvious question is whether the solution given by Corollary 4.10 meets this growth condition. If it does, then it is the unique solution that does. Suppose therefore that h and f are bounded by $|h(x)|\leq Me^{A|x|^2}$ and $|f(x,t)|\leq Me^{A|x|^2}$ on $(x,t)\in\mathbb{R}^n\times[0,T]$ for some $A>0$, $a>0$. Observe the following doubling relation for the fundamental solution

$$\Phi(x,t) = \frac{2^{n/2}}{(2\pi(2t))^{n/2}}\exp\left(-2\frac{|x|^2}{4(2t)}\right) = 2^{n/2}\Phi(x,2t)\exp\left(-\frac{|x|^2}{8t}\right).$$

For $t\leq\frac{1}{16A}=:T_0$ this implies $\Phi(x,t)\leq 2^{n/2}\Phi(x,2t)\exp(-2A|x|^2)$ We compute the first integral from the formula for the solution:

$$\begin{aligned}\left|\int_{\mathbb{R}^n}\Phi(x-y,t)h(y)\,\mathrm{d}^\mathrm{n}y\right| &\leq \int_{\mathbb{R}^n} 2^{n/2}\Phi(x-y,2t)e^{-2A|x-y|^2}Me^{A|y|^2}\,\mathrm{d}^\mathrm{n}y\\ &= 2^{n/2}M\int_{\mathbb{R}^n}\Phi(x-y,2t)e^{2A|x|^2-A|2x-y|^2}\,\mathrm{d}^\mathrm{n}y \leq 2^{n/2}Me^{2A|x|^2}\,\mathrm{d}^\mathrm{n}y.\end{aligned}$$

The last step of the calculation was achieved by the estimate $e^{-A|2x-y|^2}\leq 1$ and using the fact that for any positive time the fundamental solution has integral 1, Lemma 4.6. For the second integral of in the formula of the solution, the above estimate also applies, but further we need to integrate. Again for $t<T_0$ we have

$$\left|\int_0^t\int_{\mathbb{R}^n}\Phi(x-y,t-s)f(y,s)\,\mathrm{d}^\mathrm{n}y\,\mathrm{d}s\right| \leq \int_0^t 2^{n/2}Me^{2A|x|^2}\,\mathrm{d}s \leq 2^{n/2}Me^{2A|x|^2}T_0.$$

Together this proves that $|u(x,t)|\leq M'e^{A'|x|^2}$ on $\mathbb{R}^n\times[0,T_0]$ for $A'=2A$, $M'=2^{n/2}M(1+T_0)$ and $T_0=\frac{1}{16A}$. Thus we have proven short time unique existence for the Cauchy problem. The short time limitation is unavoidable. Consider the solution $u(x,t)=(T-t)^{-\frac{n}{2}}\exp\left(\frac{|x|^2}{4(T-t)}\right)$ of the homogeneous heat equation. It has the initial condition $h(x)=T^{-\frac{n}{2}}\exp\frac{|x|^2}{4T}$ but explodes for $t\to T$.

4.4 Heat Kernels

In the last section we proved that we found the unique (non-monstrous) solution to the Cauchy problem and proved uniqueness for Dirichlet problem. It remains to solve the Dirichlet problem, at least in some special cases. That is the goal of this section. In analogy to the Green's function of the Laplace equation we define for open subsets $\Omega \subset \mathbb{R}^n$ the heat kernel H_Ω.

Definition 4.14. *For an open domain $\Omega \subset \mathbb{R}^n$ the heat kernel $H_\Omega : \Omega \times \Omega \times \mathbb{R}^+ \to \mathbb{R}$ of Ω is characterised by the following two properties:*

(i) *For $x \in \Omega$ the function $(y,t) \mapsto H_\Omega(x,y,t) - \Phi(x-y,t)$ solves the homogeneous heat equation and extends continuously to $\bar{\Omega} \times \mathbb{R}^+_0$ with value 0 on $(y,t) \in \bar{\Omega} \times \{0\}$.*

(ii) *For $(x,t) \in \Omega \times \mathbb{R}^+$ $y \mapsto H_\Omega(x,y,t)$ extends continuously to $\bar{\Omega}$ with value 0 on $\partial\Omega$.*

We want to develop a formula for the solution similar to the Poisson representation formula. Therefore we begin by giving a version of Green's representation formula. Let u and v are two functions on $\Omega \times \mathbb{R}^+$ with appropriate regularity. Integrating Green's second formula from 0 to $T - \epsilon$ gives

$$\int_0^{T-\epsilon} \int_\Omega v(x,T-t)\triangle_x u(x,t) - \triangle_x v(x,T-t)u(x,t)\,\mathrm{d}^\mathrm{n}x\,\mathrm{d}t$$
$$= \int_0^{T-\epsilon} \int_{\partial\Omega} [v(y,T-t)\nabla_y u(y,t) - \nabla_y v(y,T-t)u(y,t)] \cdot N(y)\,\mathrm{d}\sigma(y)\,\mathrm{d}t.$$

We need a similar formula with ∂_t in place of the Laplacian so that we can combine them and get the heat operator. Therefore we take the expression we need and integrate by parts

$$\int_0^{T-\epsilon} \int_\Omega v(x,T-t)\partial_t u(x,t)\,\mathrm{d}^\mathrm{n}x\,\mathrm{d}t$$
$$= \int_\Omega v(x,T-t)u(x,t)\,\mathrm{d}^\mathrm{n}x\Big|_0^{T-\epsilon} + \int_0^{T-\epsilon} \int_\Omega \partial_t v(x,T-t)u(x,t)\,\mathrm{d}^\mathrm{n}x\,\mathrm{d}t$$
$$\int_0^{T-\epsilon} \int_\Omega v(x,T-t)\partial_t u(x,t) - \partial_t v(x,T-t)u(x,t)\,\mathrm{d}^\mathrm{n}x\,\mathrm{d}t$$
$$= \int_\Omega (u(x,T-\epsilon)v(x,\epsilon) - u(x,0)v(x,T))\,\mathrm{d}^\mathrm{n}x.$$

Combined these formulas give a Green's second formula for the heat equation. As for the Laplace equation, the next step is to set $v(y,t) = H_\Omega(x,y,t)$. Just like in that case, we must take care of the singularity it has at $t = 0$, hence the reason we only integrated up to time $T-\epsilon$. Now take $\epsilon \downarrow 0$ and use Theorem 4.7 to deduce the limit. We arrive at the following representation formula:

$$\int_0^T \int_\Omega (\dot u(y,t) - \triangle u(y,t)) H_\Omega(x,y,T-t)\,\mathrm{d}^\mathrm{n} y\,\mathrm{d}t$$
$$= \int_0^T \int_{\partial\Omega} u(z,t)\nabla_z H_\Omega(x,z,T-t)\cdot N(z)\,\mathrm{d}\sigma(z)\,\mathrm{d}t + u(x,T) - \int_\Omega u(y,0) H_\Omega(x,y,T)\,\mathrm{d}^\mathrm{n} y.$$

As with the Laplace equation, inserting the boundary conditions and inhomogeneities into this formula defines a valid solution, furnishing us with a solution to the Dirichlet problem.

Theorem 4.15 (Solution of the Dirichlet problem)**.** *Let f be a function on $\Omega \times (0,T)$, g a function on $\partial\Omega \times [0,T]$ and h a function on Ω which together with the open domain $\Omega \subset \mathbb{R}^n$ have appropriate regularity such that all appearing integrals converge absolutely. Then*

$$u(x,T) = \int_\Omega h(y) H_\Omega(x,y,T)\,\mathrm{d}^\mathrm{n} y + \int_0^T \int_\Omega f(y,t) H_\Omega(x,y,T-t)\,\mathrm{d}^\mathrm{n} y\,\mathrm{d}t$$
$$- \int_0^T \int_{\partial\Omega} g(z,t)\nabla_z H_\Omega(x,z,T-t)\cdot N(z)\,\mathrm{d}\sigma(z)\,\mathrm{d}t$$

is the unique solution of the initial and boundary value problem

$$\dot u - \triangle u = f \text{ on } \Omega\times(0,T) \qquad u = g \text{ on } \partial\Omega \times [0,T] \qquad u(x,0) = h(x) \text{ on } \Omega.$$

We do not give a proof of this statement; it is similar to the proof Poisson's representation formula 3.21. In that proof we tried to abstract out the properties that were required for the proof, particularly those of the normal derivative $\nabla G \cdot N$. The proof of this theorem works along similar lines, except we must deduce the properties from the definition of the heat kernel rather than having a concrete formula for the Green's function. Moreover the appropriate regularity conditions for f, g, h depend on the regularity of the heat kernel, which in turn depends on the domain. Let us instead prove some general properties of the heat kernel to give a taste of the task.

Lemma 4.16. *For any bounded open domain $\Omega \subset \mathbb{R}^n$ the heat kernel is unique, if it exists.*

Proof. For each $x \in \partial\Omega$ let $u(y,t) = H_\Omega(x,y,t) - \Phi(x-y,t)$. This solves the homogeneous heat equation with initial condition $h \equiv 0$ and boundary condition $u(y,t) = -\Phi(x-y,t)$ for $y \in \partial\Omega$, since $H_\Omega(x,y,t) = 0$ on the boundary. This defines a Dirichlet problem and we know that there is at most one solution, due to Theorem 4.12. □

The next property is also familiar from the Green's functions situation.

Lemma 4.17. *For all $T > 0$ and $x, y \in \bar{\Omega}$ we have $H_\Omega(x, y, T) = H_\Omega(y, x, T)$.*

Proof. We insert $u(z, t) = H_\Omega(x, z, t)$ into the above representation formula. By Theorem 4.7 (iii) and the property (ii) of the heat kernel the following integral vanishes:

$$\int_\Omega (H_\Omega(x, z, T)H_\Omega(y, z, 0) - H_\Omega(x, z, 0)H_\Omega(y, z, T)) \, \mathrm{d}^n z = H_\Omega(x, y, T) - H_\Omega(y, x, T).$$

□

This result is more important than it looks to the proof of Theorem 4.15 because we only know that the heat kernel is (for positive time) a solution to the heat equation in y. Soon we will have a regularity result about solutions to the heat equation. Symmetry then lets us transfer that regularity to x, which is the variable we need to differentiate to prove u solves the heat equation.

Lemma 4.18. *For any bounded connected open domain $\Omega \subset \mathbb{R}^n$ the corresponding heat kernel is positive on the corresponding parabolic cylinder, if it exists.*

Proof. The fundamental solution $\Phi(x, t)$ is positive on $(x, t) \in \mathbb{R}^n \times \mathbb{R}^+$. For bounded open domains $\Omega \subset \mathbb{R}^n$ and given $x \in \Omega$ the difference $\Phi(x - y, t) - H_\Omega(x, y, t)$ of the fundamental solution minus the heat kernel is the unique solution of the heat equation on $\Omega \times [0, T]$ which vanishes on $\Omega \times \{t = 0\}$ and coincides on $\partial\Omega \times [0, T]$ with $\Phi(x - y, t)$. This solution is for all $\epsilon > 0$ on $\Omega \times \{t = \epsilon\}$ and on $\partial\Omega \times [0, T]$ not larger than $\Phi(x - y, t)$. By the Maximum Principle it is not larger than $\Phi(x - y, t)$ and $H_\Omega(x, y, t)$ is positive. □

4.5 Heat Kernel of $(0, 1)$

Despite our hard work, we still haven't actually solved the Dirichlet problem for even a single domain Ω. It is long past time to rectify that. We begin with the simplest case $n = 1$ where every open bounded domain is the union of intervals. Up to scaling and translation then, we need only consider the unit interval $(0, 1)$.

There are several ideas that lead to the heat kernel. The method of images will be explored in the exercises. Here we give an argument based on the eigenfunctions. If you recall from the beginning of the chapter, the special class of separable solutions is connected to the eigenfunctions of the Laplacian $-\triangle$. In dimension one the eigenfunctions $e^{\pm 2\pi i|k|x}$ have eigenvalues $4\pi^2|k|^2$. If we look for eigenfunctions that vanish on the boundary, then this is only possible if $k \in \frac{1}{2}\mathbb{Z}$ and then

$$h_k(x) = \sqrt{2}\sin 2\pi kx$$

is the unique solution up to scaling. This particular scaling has been chosen because it makes these functions orthonormal with respect to the inner product on $L^2([0,1])$. Due to the Stone-Weierstrass theorem, these functions are also dense in the space of functions that vanish at $x = 0, 1$. But by Property (ii) of heat kernels, $H_{(0,1)}$ is such a function. Therefore we expect

$$H_{(0,1)}(x,y,t) = \sum_{k \in \frac{1}{2}\mathbb{N}^+} a_k(x,t) h_k(y).$$

This is essentially the Fourier series of the heat kernel. The unique solution to the homogeneous heat equation with h_k as initial condition and vanishing for $x = 0, 1$ is

$$u_k(x,t) = e^{-4\pi^2 k^2 t}\sqrt{2}\sin 2\pi k x.$$

If $H_{(0,1)}$ is the heat kernel of $(0,1)$ then it must fulfil the representation for these functions. Hence

$$u_l(x,t) = \int_{\mathbb{R}^n} H_{(0,1)}(x,y,t) h_l(y)\,\mathrm{d}^n y + 0 + 0 = \sum_{k \in \frac{1}{2}\mathbb{N}^+} a_k(x,t) \int_{\mathbb{R}^n} h_k(y) h_l(y)\,\mathrm{d}^n y = a_l(x,t).$$

This brings us to a formula for the heat kernel

$$H_{(0,1)}(x,y,t) = \sum_{k \in \frac{1}{2}\mathbb{N}^+} u_l(x,t) h_k(y) = \sum_{n=1}^{\infty} 2e^{-\pi^2 n^2 t} \sin(\pi n x)\sin(\pi n y).$$

The method of images leads to the formula

$$H_{(0,1)}(x,y,t) = \frac{1}{2}\Theta(\tfrac{x-y}{2}, \pi i t) - \frac{1}{2}\Theta(\tfrac{x+y}{2}, \pi i t)$$

where $\Theta(x,\tau)$ is Jacobi's Theta function, a well-studied 'special' function defined by the series

$$\Theta(x,\tau) = \sum_{k \in \mathbb{Z}} e^{2\pi i k x + \pi i \tau k^2}.$$

This sum converges on the domain $(x,\tau) \in \mathbb{C} \times \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ very rapidly since $e^{\pi i \tau k^2}$ decays exponentially with respect to k^2, making it useful for computation. The sine formula for the heat kernel also has this property, but none-the-less it is useful to be able to call on standard functions when using a program such as Mathematica or Matlab. The Theta function is theoretically important because of its quasiperiodicity:

$$\Theta(x+1,\tau) = \Theta(x,\tau), \qquad \Theta(x+\tau,\tau) = \Theta(x,\tau)e^{-\pi i \tau - 2\pi i x}.$$

From the heat kernel on $(0,1)$ we can construct the heat kernel on any interval. The fundamental solution scales according to $\Phi(x-y,t) = \frac{1}{r^n}\Phi(\frac{x}{r} - \frac{y}{r}, \frac{t}{r^2})$. It is also invariant if we translate x and y by the same amount. Since the heat kernel is unique, it must be

$$H_{(a,b)}(x,y,t) = \frac{1}{b-a} H_{(0,1)}\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}, \frac{t}{(b-a)^2}\right).$$

This even gives us a heat kernel on cubes in $\mathbb{R}^n$, since the heat kernel of the Cartesian product of two domains can be easily calculated in terms of the heat kernels of both domains:

Lemma 4.19. *If $\Omega \subset \mathbb{R}^m$ and $\Omega' \subset \mathbb{R}^n$ are two open, bounded and connected domains with given heat kernels H_Ω and $H_{\Omega'}$, then the heat kernel of $\Omega \times \Omega'$ is given by*

$$H_{\Omega\times\Omega'}((x,x'),(y,y'),t) = H_\Omega(x,y,t)H_{\Omega'}(x',y',t) \quad (x,x'),(y,y') \in \bar{\Omega}\times\bar{\Omega}' \quad t \in \mathbb{R}^+.$$

Proof. For any $(x,x',t) \in \Omega\times\Omega'\times\mathbb{R}^+$ the function $(y,y') \mapsto H_\Omega(x,y,t)H_{\Omega'}(x',y',t)$ extends by the value zero continuously to $\partial(\Omega\times\Omega') = (\partial\Omega\times\Omega')\cup(\Omega\times\partial\Omega')$. The Laplace operator of the Cartesian product is the sum of the corresponding Laplace operators. We calculate

$$\begin{aligned}\partial_t(H_\Omega H_{\Omega'}) - (\triangle_y + \triangle_{y'})H_\Omega H_{\Omega'} &= (\partial_t H_\Omega)H_{\Omega'} + H_\Omega(\partial_t H_{\Omega'}) - (\triangle_y H_\Omega)H_{\Omega'} - H_\Omega(\triangle_{y'} H_{\Omega'})\\ &= (\partial_t H_\Omega - \triangle_y H_\Omega)H_{\Omega'} + H_\Omega(\partial_t H_{\Omega'} - \triangle_{y'} H_{\Omega'}) = 0.\end{aligned}$$

Hence for all $(x,x') \in \Omega\times\Omega'$ the function $(y,y',t) \mapsto H_\Omega(x,y,t)H_{\Omega'}(x',y',t)$ solves the homogeneous heat equation. The product of both fundamental solutions is the fundamental solution on $\mathbb{R}^{m+n}$. Hence for all $(x,x') \in \Omega\times\Omega'$ the function

$$\begin{aligned}(y,y',t) \mapsto & H_\Omega(x,y,t)H_{\Omega'}(x',y't) - \Phi(x-y,t)\Phi(x'-y',t)\\ &= [H_\Omega - \Phi(x-y,t)][H_{\Omega'} - \Phi(x'-y',t)]\\ &\quad + \Phi(x-y,t)[H_{\Omega'} - \Phi(x'-y',t)] + [H_\Omega - \Phi(x-y,t)]\Phi(x'-y',t)\end{aligned}$$

extends continuously to $\bar{\Omega}\times\bar{\Omega}'\times\mathbb{R}_0^+$ by setting it zero on $(y,y',t) \in \bar{\Omega}\times\bar{\Omega}'\times\{0\}$. □

The minor technicality is that the boundaries of the Cartesian products $(0,1)^n \subset \mathbb{R}^n$ are not continuously differentiable submanifolds of $\mathbb{R}^n$ and our proof of the divergence theorem does not apply to these Cartesian products. However, the divergence theorem does hold for these Cartesian products, so this is indeed only a technicality.

We close this chapter with a final result on regularity. Due to the existence of monster solutions, we cannot hope for analyticity in the time coordinate, but we at least have smoothness.

Corollary 4.20. *Any solution u of the homogeneous heat equation on an open domain $\Omega \subset \mathbb{R}^n\times\mathbb{R}$ is smooth and for fixed t analytic with respect to x.*

Proof. For any point in the domain, we can find a small cube in space and time that contains the point. By translation, assume that the cube is $[0,r]^n\times[0,T]$ and the point is time T. Then using the heat kernel on this domain, we obtain from the representation formula

$$u(x,T) = -\int_0^T\int_{\partial[0,r]^n} u(z,t)\nabla_z H_{[0,r]^n}(x,z,T-t)\cdot N(z)\,\mathrm{d}\sigma(z)\,\mathrm{d}t + \int_{[0,r]^n} u(y,0)H_{[0,r]^n}(x,y,T)\,\mathrm{d}^n y.$$

It remains to show that the regularity of the heat kernel it transferred to u. This can be calculated using the explicit formula, but we give a more conceptual argument. In the proof of Theorem 4.7 we show that $\Phi(x-y,t)$ converges on the complement of $y \in B(x,\delta)$ uniformly to zero in the limit $t \downarrow 0$. The same is true for all partial derivatives and due to condition (ii) in Definition 4.14 also for $H_{(0,1)^n}(x,y,t)$. By Lemma 4.17 the integral for $u(x,T)$ is smooth at all $x \in (0,r)^n$. For $(z,t) \in \partial(0,r)^n \times [0,T]$ the Taylor series of $x \mapsto H_{[0,r]^n}(x,z,T-t)$ converges uniformly on compact subsets of $x \in (0,r)^n$ to $H_{[0,r]^n}(x,z,T-t)$. □

Chapter 5

Wave Equation

The wave equation describes phenomena which propagate with finite speed through space time. The example of sound and electrodynamic (light) waves motivated the investigation of this equation in $n = 3$, though it is also a useful model of vibrating strings and drums in $n = 1$ and $n = 2$ respectively. Later these methods were generalised to non-linear hyperbolic equations in order to describe gravitational waves.

In this final chapter we consider the homogeneous and inhomogeneous wave equation on open subsets of $\mathbb{R}^n \times \mathbb{R}$ for $n \leq 3$. In particular we study the Cauchy problem for $t > 0$

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = f \qquad \text{on} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \qquad \text{with}$$
$$u(x,0) = g(x) \qquad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = h(x).$$

The wave equation is a linear second order PDE. The coefficient matrix for the second derivatives has one positive and n negative eigenvalues and is neither definite nor semi-definite. In the second chapter we introduced this differential equation as the simplest hyperbolic differential equation. The general theory of hyperbolic equations is distinctly different to that of elliptic and parabolic equations.

We see for the Cauchy problem that we have given not only the value of u on the initial boundary but also its normal derivative. The intuition is that if you choose a point $(x_0, 0)$ then $\triangle u(x_0, 0) = \triangle g(x_0)$. Thus $\partial_t^2 u(x_0, 0)$ can be determined from the PDE but not $\partial_t u(x_0, 0)$. The simple example of the linear functions $u(x,t) = at + b$ show that these two values are indeed independent. Conversely, for smooth functions f, g, h these initial conditions are sufficient to determine all derivatives on u at $(x_0, 0)$. For example

$$\partial_t^3 u(x,0) = \triangle \partial_t u(x,0) + \partial_t^3 f(x,0) = \triangle h(x) + \partial_t^k f(x,0),$$
$$\partial_t^4 u(x,0) = \triangle \partial_t^2 u(x,0) + \partial_t^4 f(x,0) = \triangle^2 g(x) + \triangle f(x,0) + \partial_t^4 f(x,0).$$

This discussion may remind you of Definition 1.6 of characteristic and non-characteristic curves. Let's make a brief detour to see how the method of characteristics can be generalised to the wave equation for $n = 1$. Consider a path $(x(s), t(s))$ in the domain. Let us consider how the three functions u, $v = \partial_t u$, $w = \partial_x u$ behave along such a curve. We use a dot for derivative with respect to s. By the chain rule $\dot{u} = v\dot{t} + w\dot{x}$. The derivative for v and w are similar to one another.

$$\dot{v} = \partial_t v\dot{t} + \partial_x v\dot{x}, \quad \dot{w} = \partial_t w\dot{t} + \partial_x w\dot{x}.$$

We need to relate these in such a way that we remove the direct dependence on x and t. The equality of partial derivatives implies $\partial_x v = \partial_t w$ and from the wave equation we have $\partial_t v - \partial_x w = 0$. Substitution shows us that

$$\dot{v} = \partial_t v\dot{t} + \partial_x v\dot{x}, \quad \dot{w} = \partial_x v\dot{t} + \partial_t v\dot{x}.$$

So we can equate these two expressions if $\dot{x} = \dot{t} = 1$ or $\dot{x} = -\dot{t} = -1$. Thus there are two characteristics through every point. Unlike for crossing characteristics in first order systems, this is not necessarily a problem. On the characteristic $x - t = c$ we have the system of ODEs

$$\dot{u} = v + w, \quad \dot{v} - \dot{w} = 0.$$

And on the characteristic $x + t = c$ we have

$$\dot{\tilde{u}} = \tilde{v} - \tilde{w}, \quad \dot{\tilde{v}} + \dot{\tilde{w}} = 0.$$

The tildes indicate that these functions are on different curves We see that both systems are underdetermined (three unknowns, two equations) so there is the possibility that they can be made to agree everywhere. The method of characteristics for higher order PDEs leads to the celebrated theorem of Cauchy and Kowalevski (also spelt Kovalevskaya) on the existence of PDEs with analytic coefficients. We do not pursue this line of inquiry further, nor shall we use Fourier analysis to solve the wave equation, though both methods work well. Instead we will use a classical method that links back to the first chapter. Hopefully the above digression has provided some deeper insight as to why the classical method works.

5.1 D'Alembert's Formula

First we solve the Cauchy problem in one dimension (of space). We may factorise the wave operator (also called D'Alembert's operator)

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right).$$

If u solves the homogeneous wave equation, then $v(x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t)$ solves $\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0$. This is the transport equation with constant coefficient with the unique solution

$$v(x,t) = a(x-t) \quad \text{with} \quad a(x) = v(x,0).$$

So the solution $u(x,t)$ of the wave equation solves the first order linear PDE

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = a(x-t).$$

This is an inhomogeneous transport equation with constant coefficients with the solution

$$u(x,t) = b(x+t) + \int_0^t a(x + (t-s) - s)\,\mathrm{d}s = b(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(y)dy$$

with $b(x) = u(x,0)$. The initial values $u(x,0) = g(x)$ and $\frac{\partial u}{\partial t}(x,0) = h(x)$ yields

$$b(x) = g(x) \quad \text{and} \qquad a(x) = v(x,0) = \frac{\partial u}{\partial t}(x,0) - \frac{\partial u}{\partial x}(x,0) = h(x) - g'(x).$$

If we insert this in our solutions, then we obtain

$$u(x,t) = g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} \left(h(y) - g'(y)\right) dy$$

Hence the solution of the initial value problem of the wave equation is given by

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.$$

Moreover, this must be the unique solution, since the transport equation has a unique solution. In summary

Theorem 5.1 (D'Alembert's Formula)**.** *If $g : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and $h : \mathbb{R} \to \mathbb{R}$ continuously differentiable, then*

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$$

is a twice continuously differentiable function on $\mathbb{R} \times \mathbb{R}_0^+$ that is the unique solution of the Cauchy problem of the homogeneous wave equation.

First an observation on the regularity. If solution is k-times differentiable, if g and H are k times differentiable, or equivalently if g is k times differentiable and h is $(k-1)$ times differentiable. So the regularity of the solution does not improve with time, as it does for solutions of the heat equation.

We interpret the fact that the value of the solution at (x,t) depends only on the values of g at $x \pm t$ and the values of h at points in the interval $[x-t, x+t]$ as a bound of 1 on the speed of propagation, since the trajectories from these points to (x,t) propagate with speed not larger than 1. A stronger statement is possible. Using an antiderivative of h then we can write

$$u(x,t) = F(x+t) + G(x-t).$$

Conversely, every function of this form is a solution of the wave equation if F and G are twice differentiable (Exercise). Hence the value $u(x,t)$ of the solution at (x,t) depends only on the values of F and G at $x \pm t$ and the propagation speed is exactly 1. We call this the decomposition into forward and backward travelling waves.

5.2 Solution on the half-line

While we are mainly interested in the Cauchy problem on $\mathbb{R}^n$, in one dimension it is straightforward to reflect and derive the solution on the half-line. We will need this solution later in the chapter. Stated precisely, we solve the following problem.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \qquad u(0,t) = 0 \qquad \text{for} \quad t \in \mathbb{R}_0^+,$$

$$u(x,0) = g(x) \quad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = h(x) \qquad \text{for} \quad x \in \mathbb{R}^+.$$

The trick is to extend the functions g and h to odd functions on the whole space $\mathbb{R} \times \mathbb{R}_0^+$ by a reflection:

$$\tilde{g}(x) = \begin{cases} g(x) & \text{for } x \geq 0, \\ -g(-x) & \text{for } x \leq 0, \end{cases} \qquad \tilde{h}(x) = \begin{cases} h(x) & \text{for } x \geq 0, \\ -h(-x) & \text{for } x \leq 0. \end{cases}$$

For any solution $\tilde{u}$ of the initial value problem

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} = 0 \qquad \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

$$\tilde{u}(x,0) = \tilde{g}(x) \quad \text{and} \qquad \frac{\partial \tilde{u}}{\partial t}(x,0) = \tilde{h}(x) \qquad \text{for} \quad x \in \mathbb{R},$$

the function $(x,t) \mapsto -\tilde{u}(-x,t)$ is also solution. Due to the uniqueness of the solution both solutions coincide: $\tilde{u}(-x,t) = -\tilde{u}(x,t)$. By this argument we conclude that $\tilde{u}$ is an odd function and $\tilde{u}(0,t) = 0$. Hence this solution restricts to give a solution of the half-line problem.

Conversely, if we take a solution to the half line problem, one can check that its reflected extension solves the Cauchy problem on $\mathbb{R}^n$. The important point is the check the first and second derivatives of the reflection exist at $x = 0$, but this is guaranteed by the fact that u vanishes there. Therefore there is a bijection between solutions of the two problems and in particular the solution on the half-line is unique.

Explicitly the solution on the half-line is given by

$$u(x,t) = \begin{cases} \frac{1}{2}\left(g(x+t) + g(x-t) + \int_{x-t}^{x+t} h(y)dy\right) & \text{for } 0 \le t \le x \\ \frac{1}{2}\left(g(t+x) - g(t-x) + \int_{t-x}^{t+x} h(y)dy\right) & \text{for } 0 \le x \le t. \end{cases}$$

Note that the waves propagating towards the boundary at $x = 0$ are reflected at the boundary and propagate back.

5.3 Spherical Means of the Wave Equation

When we studied the Laplace equation, we saw that the spherical means of its solutions (harmonic functions) did not depend on the radius of the sphere. The spherical means of solutions of the wave equation do depend on the radius of the sphere, but in a controlled way. In fact they obey a PDE! This PDE is similar to the one-dimensional wave equation. This opens an avenue to solve the initial value problem of the wave equation in any odd dimension, though for this course we will stick to $n = 3$. We define for all $x \in \mathbb{R}^n, t \geq 0, r > 0$ the spatial-spherical mean

$$\mathcal{S}[u](x,r,t) := \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y,t)\, \mathrm{d}\sigma(y).$$

Here t is treated as an additional parameter and not integrated. With this understanding we reuse the same notation for the spherical means. For brevity we define $U(x,r,t) = \mathcal{S}[u](x,r,t)$, $G(x,r) = \mathcal{S}[g](x,r)$, and $H(x,r) = \mathcal{S}[h](x,r)$.

Lemma 5.2. *If $u \in C^m(\mathbb{R}^n \times \mathbb{R}_0^+)$ is a m-times continuously differentiable solution of the initial value problem (with continuous partial derivatives of order $\le m$ on $\mathbb{R}^n \times \mathbb{R}_0^+$) of the Cauchy problem of the homogeneous wave equation. The spherical mean $U(x,r,t)$ for fixed $x \in \mathbb{R}^n$ is an m-times differentiable function on $(r,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, which solves the following initial value problem of the Euler-Poisson-Darboux Equation (with continuous partial derivatives of order $\le m$):*

$$\frac{\partial^2 U}{\partial t^2}(x,r,t) - \frac{\partial^2 U}{\partial r^2}(x,r,t) - \frac{n-1}{r}\frac{\partial U}{\partial r}(x,r,t) = 0 \quad \textit{on} \quad (r,t) \in \mathbb{R}^+ \times \mathbb{R}^+$$

$$U(x,r,0) = G(x,r) \quad \textit{and} \quad \frac{\partial U}{\partial t}(x,r,0) = H(x,r)$$

Proof. By a substitution the domain of the integral becomes independent of t and r:

$$U(x,r,t) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} u(ry + x, t)\, \mathrm{d}\sigma(y).$$

Hence we may calculate the derivative

$$\begin{aligned}\frac{\partial U}{\partial r}(x,r,t) &= \frac{1}{n\omega_n}\int_{\partial B(0,1)} \nabla u(x+ry,t)\cdot y\,\mathrm{d}\sigma(y)\\ &= \frac{r}{n\omega_n}\int_{B(0,1)} \triangle u(x+ry,t)\,\mathrm{d}^{\mathrm{n}}y = \frac{r}{n\omega_n r^n}\int_{B(x,r)} \triangle u(y,t)\,\mathrm{d}^{\mathrm{n}}y.\end{aligned}$$

In the limit at $r \downarrow 0$, we can recognise the last expression as $\frac{r}{n}$ multiplied with the globular mean of $\triangle u$. The mean has a limit $\triangle u(x,r)$, which implies $\lim_{r\to 0}\frac{\partial U}{\partial r}(x,r,t) = 0$. Differentiating further we get

$$\begin{aligned}\frac{\partial^2 U}{\partial r^2}(x,r,t) &= \frac{\partial}{\partial r}\left(\frac{1}{n\omega_n r^{n-1}}\int_0^r\int_{\partial B(x,s)} \triangle u(y,t)\,\mathrm{d}^{\mathrm{n}}y\,\mathrm{d}s\right)\\ &= \frac{1-n}{n\omega_n r^n}\int_{B(x,r)} \triangle u(y,t)\,\mathrm{d}^{\mathrm{n}}y + \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(x,r)} \triangle u(y,t)\,\mathrm{d}\sigma(y)\\ &= \frac{1-n}{r}\frac{\partial U}{\partial r}(x,r,t) + \mathcal{S}[\triangle u](x,r,t).\end{aligned}$$

Finally, we use the wave equation to change this last term.

$$\mathcal{S}[\triangle u](x,r,t) = \mathcal{S}[\partial_t^2 u](x,r,t) = \partial_t^2\mathcal{S}[u](x,r,t) = \frac{\partial^2 U}{\partial t^2}(x,r,t).$$

□

5.4 Solution in Dimension 3

We shall see that for odd dimensions the spherical means of solutions of the wave equation can be transformed into solutions of the one-dimensional wave equation, but not for even dimensions. For this reason we shall next solve the initial value problem of the wave equation in three dimensions. In this section we consider for any $x \in \mathbb{R}^3$ the following initial value problem for the spherical means of a solution of the wave equation:

$$\begin{aligned}\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial r^2} - \frac{2}{r}\frac{\partial U}{\partial r} &= 0 \quad \text{on} \quad (x,r,t) \in \{x\}\times\mathbb{R}^+\times\mathbb{R}^+\\ U = G \quad \text{and} \quad \frac{\partial U}{\partial t} &= H \quad \text{on} \quad (x,r,t) \in \{x\}\times\mathbb{R}^+\times\{0\}.\end{aligned}$$

The substitution $\tilde{U} = rU$ transforms the above into the following:

$$\frac{\partial^2\tilde{U}}{\partial t^2} - \frac{\partial^2\tilde{U}}{\partial r^2} = 0 \text{ on } (x,r,t) \in \{x\}\times\mathbb{R}^+\times\mathbb{R}^+, \qquad \tilde{U}(x,0,t) = 0 \text{ for } t \in \mathbb{R}_0^+,$$

$$\tilde{U}(x,r,0) = \tilde{G}(x,r) = rG(x,r) \;\text{ and }\; \frac{\partial\tilde{U}}{\partial t}(x,r,0) = \tilde{H}(x,r) = rH(x,r) \text{ for } r \in \mathbb{R}^+.$$

We solved this initial value problem in the Section 5.2. The solution is

$$\tilde{U}(x,r,t) = \frac{1}{2}\Big(\tilde{G}(x,r+t) - \tilde{G}(x,t-r)\Big) + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(x,s)\,\mathrm{d}s \quad \text{for } 0 \le r \le t.$$

But this isn't what we wanted. We wanted the solve the wave equation. Thus we must undo all the transforms and recover u. The continuity of $u(x,t)$ implies

$$u(x,t) = \lim_{r\downarrow 0} U(x,r,t) = \lim_{r\downarrow 0} \frac{\tilde{U}(x,r,t)}{r}.$$

We compute this limit for each part of the formula of $\tilde{U}$.

$$\begin{aligned}\lim_{r\downarrow 0}\frac{1}{2r}\Big(\tilde{G}(x,r+t) - \tilde{G}(x,t-r)\Big) &= \lim_{r\downarrow 0}\tfrac{1}{2}\Big(\tfrac{\tilde{G}(x,t+r)-\tilde{G}(x,t)}{r} + \tfrac{\tilde{G}(x,t-r)-\tilde{G}(x,t)}{-r}\Big)\\ = \frac{\partial \tilde{G}(x,t)}{\partial t} = \frac{\partial}{\partial t}\left(t\mathcal{S}[g](x,t)\right) &= \mathcal{S}[g](x,t) + \frac{t}{4\pi t^2}\int_{\partial B(x,t)} \nabla u(y)\cdot N\,\mathrm{d}\sigma(y)\end{aligned}$$

using Equation (3.4), and

$$\lim_{r\downarrow 0}\frac{1}{2r}\int_{-r+t}^{r+t} \tilde{H}(x,s)\,\mathrm{d}s = \tilde{H}(x,t) = t\mathcal{S}[h](x,t)$$

Therefore we obtain for all $x \in \mathbb{R}^3, t > 0$

$$u(x,t) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} \Big(th(y) + g(y)\Big)\,\mathrm{d}\sigma(y) + \frac{1}{4\pi t^2}\int_{\partial B(x,t)} \nabla_y g(y)\cdot(y-x)\,\mathrm{d}\sigma(y)$$

using the fact that $tN(y) = y - x$ for points $y \in \partial B(x,t)$. The is Kirchhoff's Formula for the solution of the initial value problem of the three dimensional wave equation. We see, like the one dimensional wave equation, that for the three dimensional wave equation the value at (x,t) only depends on the values $(y,0)$ for $y \in \partial B(x,t)$. We again stylise this fact to mean that all waves travel at speed 1.

5.5 Solution in Dimension 2

In two dimensions the Euler-Poisson-Darboux equations cannot be transformed into the one-dimensional wave equation. We present another method, the method of descent, and transform the initial value problem of the two-dimensional wave equation into a special type of initial value problem of the three-dimensional wave equation: We choose initial values which depend only on the coordinates x_1 and x_2 and not on the coordinate x_3. If g, h are the initial values of the 2-dimensional problem, let

$$\bar{g}(x_1,x_2,x_3) = g(x_1,x_2), \quad \bar{h}(x_1,x_2,x_3) = h(x_1,x_2).$$

By the previous section, we know how to calculate the solution $\bar{u}(x,t)$ on $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$ of the initial value problem

$$\frac{\partial^2 \bar{u}(x,t)}{\partial t^2} - \triangle \bar{u}(x,t) = 0 \qquad \text{for} \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$$
$$\bar{u}(x,0) = \bar{g}(x) \text{ and } \quad \frac{\partial \bar{u}}{\partial t}(x,0) = \bar{h}(x) \qquad \text{for} \quad x \in \mathbb{R}^3.$$

We observe that if a function f does not depend x_3 then the mean of that function over $\partial B(x,r)$ also does not depend on x_3:

$$\frac{\partial}{\partial x_3}\mathcal{S}[f](x,r) = \frac{\partial}{\partial x_3}\frac{1}{n\omega_n r^{n-1}}\int_{\partial B(0,r)} f(x+y)\,\mathrm{d}\sigma(y) = \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(0,r)} 0\,\mathrm{d}\sigma(y) = 0.$$

The solution $\bar{u}$ is given by Kirchhoff's formula. The second expression in that formula is not quite a spherical mean, because the integrand also depends on x. We need to check it directly

$$\frac{\partial}{\partial x_3}\int_{\partial B(x,t)} \nabla_y g(y)\cdot(y-x)\,\mathrm{d}\sigma(y) = \frac{\partial}{\partial x_3}\int_{\partial B(0,t)} \nabla_y g(x+y)\cdot y\,\mathrm{d}\sigma(y) = 0.$$

Together this shows that $\bar{u}$ does not depend on x_3. If we define $u(x_1,x_2,t) = \bar{u}(x_1,x_2,0,t)$ then

$$(\partial_t^2 - \triangle_{\mathbb{R}^2})u = (\partial_t^2 - \triangle_{\mathbb{R}^2})\bar{u} - \frac{\partial^2 \bar{u}}{\partial x_3^2} = (\partial_t^2 - \triangle_{\mathbb{R}^3})\bar{u} = 0.$$

Hence we have found a solution to the two dimensional wave equation. The initial conditions are clear. The choice of $x_3 = 0$ is not important; $\bar{u}$ is constant in x_3 so any other choice gives the same function.

Let's try to use Kirchhoff's formula but remove any mention of x_3. We use the notation $\bar{x} = (x_1,x_2)$ when $x = (x_1,x_2,x_3) \in \mathbb{R}^3$. We need to integrate over spheres. The height function $\gamma(z) = \sqrt{r^2 - |z-\bar{x}|^2}$ on the two-dimensional ball $z \in B(\bar{x},r)$ yields by the formula $\Psi(z) = (z, \pm\gamma(z))$ a parametrisations of both hemispheres of the boundary of the three-dimensional ball $B\left((\bar{x},0),r\right)$ by the two-dimensional ball $B(\bar{x},r)$. The two hemispheres do not cover $\partial B((\bar{x},0),r)$ completely, but the missing equator is one-dimensional and has measure zero with respect to $\mathrm{d}\sigma(y)$. We have already made some calculations for parametrisations that are graphs after Lemma 2.6, and using those formulas here gives

$$\sqrt{\det(\Psi'(z))^T\Psi(z)} = \sqrt{1+(\nabla\gamma(z))^2} = \frac{r}{\sqrt{r^2-|z-\bar{x}|^2}}.$$

By the definition of integration over a submanifold:

$$\int_{\partial B((\bar{x},0),r)} \bar{g}(y)\,\mathrm{d}\sigma(y) = 2\int_{B(\bar{x},r)} g(z)\sqrt{1+(\nabla\gamma(z))^2}\,\mathrm{d}^2 z = 2r\int_{B(\bar{x},r)} \frac{g(z)}{\sqrt{r^2-|z-\bar{x}|^2}}\,\mathrm{d}^2 z.$$

This gives finally the following formula for $u(\bar{x},t)$ on $(\bar{x},t)\in\mathbb{R}^2\times\mathbb{R}^+$:

$$\begin{aligned} u(\bar{x},t) &= \frac{1}{4\pi t^2}\int_{\partial B((\bar{x},0),r)}\Big(t\bar{h}(y)+\bar{g}(y)+\nabla_y\bar{g}(y)\cdot(y-x)\Big)\,\mathrm{d}\sigma(y)\\ &= \frac{1}{2\pi t}\int_{B(\bar{x},t)}\frac{th(z)+g(z)+\nabla g(z)\cdot(z-\bar{x})}{\sqrt{t^2-|z-\bar{x}|^2}}\,\mathrm{d}^2z. \end{aligned}$$

This formula also carries the name Poisson's formula. It shows that in two dimensions the propagation speed is bounded by 1.

This method of deriving the solution of the initial value problem in a lower dimension by transforming the initial value problem into an initial value problem in the higher dimensional space, is called the method of descent. Here the initial values do not depend on some of the coordinates of the higher dimensional space. Ponder this: can we obtain the solution of the one-dimensional wave equation by this method of descent from Poisson's formula?

5.6 Inhomogeneous Wave Equation

We have seen in exercises how Duhamel's principle can use the solution of the initial value problem of a homogeneous time-evolution equation to solve the inhomogeneous equation. It also applies to the wave equation, after we put it into the appropriate form: a first order linear ODE on the function space consisting of pairs of functions on $x\in\mathbb{R}^n$:

$$\frac{d}{dt}\begin{pmatrix}u(\cdot,t)\\ \frac{\partial u}{\partial t}(\cdot,t)\end{pmatrix}=\begin{pmatrix}0 & 1\\ \triangle & 0\end{pmatrix}\begin{pmatrix}u(\cdot,t)\\ \frac{\partial u}{\partial t}(\cdot,t)\end{pmatrix}+\begin{pmatrix}0\\ f(\cdot,t)\end{pmatrix}.$$

In accordance with the principle we try calculate the special solution of the inhomogeneous wave equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}-\triangle u=f && && \text{for}\quad (x,t)\in\mathbb{R}^n\times\mathbb{R}^+\\ &\quad u(x,0)=0\quad\text{and} && \frac{\partial u}{\partial t}(x,0)=0 && \text{for}\quad x\in\mathbb{R}^n \end{aligned}$$

via the family of solutions of the homogeneous wave equation whose initial values is given by the inhomogeneity. Suppose $u(x,t,s)$ solves

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}-\triangle u=0 && && \text{for}\quad (x,t)\in\mathbb{R}^n\times(s,\infty)\\ &u(x,s,s)=0\quad\text{and} && \frac{\partial u}{\partial t}(x,s,s)=f(x,s) && \text{for}\quad x\in\mathbb{R}^n, \end{aligned}$$

for any $s \in \mathbb{R}^+$, then $u(x,t) = \int_0^t u(x,t,s)\,\mathrm{d}s$ solves the former inhomogeneous wave equation since

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial}{\partial t}\left(u(x,t,t) + \int_0^t \frac{\partial u}{\partial t}(x,t,s)\,\mathrm{d}s\right) = \frac{\partial}{\partial t}\int_0^t \frac{\partial u}{\partial t}(x,t,s)\,\mathrm{d}s =$$
$$= \frac{\partial u}{\partial t}(x,t,t) + \int_0^t \frac{\partial^2 u}{\partial t^2}(x,t,s)\,\mathrm{d}s = f(x,t) + \int_0^t \triangle u(x,t,s)\,\mathrm{d}s = f(x,t) + \triangle u(x,t).$$

Consequently the initial value problem of the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = f \qquad \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$$
$$u(x,0) = g(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = h(x) \qquad \text{for} \quad x \in \mathbb{R}^n$$

is the sum of the former special solution with trivial initial value and the solution of the corresponding homogeneous initial value problem.

Finally we investigate how the presence determines the past. The wave equations is invariant with respect to time translation and reversal $t \mapsto T - t$. However, this transformation replaces $\frac{\partial u}{\partial t}$ by $-\frac{\partial u}{\partial t}$. Therefore the values $u(x,t)$ of the solution of the final value problem

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = f \qquad \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^-$$
$$u(x,T) = g(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,T) = h(x) \qquad \text{for} \quad x \in \mathbb{R}^n$$

are given by the values $u(x, T-t)$ of the solution of the initial value problem with initial values g and $-h$ and inhomogeneity $(x,t) \mapsto f(x, T-t)$. This means that we can derive both the future and the past from the presence. Both solutions fit together and form a solution $u(x,t)$ of the wave equation on $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ which is completely determined by its values $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$ on $x \in \mathbb{R}^n$.

5.7 Energy Methods

Unlike elliptic and parabolic PDES, hyperbolic PDEs do not satisfy a maximum principle. The key idea of the maximum principle was the connection between the differential operator and the Hessian, to control where extrema can occur. The case that the Hessian is never definite is exactly the elliptic PDEs and their limiting cases as the parabolic PDEs (Theorems 3.13 and 4.11). Our calculations above however suggest that solutions of the Cauchy problem are unique, and indeed they are. Thus we must seek a new method to prove this. The class of techniques we are about to see go by the name “energy methods” due to their inspiration from physics.

Theorem 5.3 (Uniqueness of the solutions of the wave equation)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the following initial values problem of the wave equation*

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \triangle u &= f && \text{on} \quad \Omega \times (0,T) \\ u(x,t) &= g(x,t) && \text{on} \quad \Omega \times \{t=0\} \qquad \text{and on} \quad \partial\Omega \times (0,T) \\ \frac{\partial u}{\partial t}(x,0) &= h(x) && \text{on} \quad \Omega \times \{t=0\} \end{aligned}$$

has a unique solution in $C^2(\Omega \times (0,T))$ with continuous extensions of $\partial^\alpha u$ to $\bar{\Omega} \times [0,T]$ for $|\alpha| \leq 2$.

Proof. The difference of two solutions solves the analogous homogeneous initial value problem with $f = g = h = 0$. For such a solution we define the energy as

$$e(t) = \frac{1}{2} \int_\Omega \left(\left(\frac{\partial u}{\partial t}(x,t) \right)^2 + |\nabla u(x,t)|^2 \right) \mathrm{d}^{\mathrm{n}} x.$$

Then we calculate

$$\begin{aligned} \frac{de}{dt}(t) &= \int_\Omega \left(\frac{\partial^2 u}{\partial t^2}(x,t) \frac{\partial u}{\partial t}(x,t) + \frac{\partial \nabla u}{\partial t} u(x,t) \nabla u(x,t) \right) \mathrm{d}^{\mathrm{n}} x \\ &= \int_\Omega \frac{\partial u}{\partial t}(x,t) \left(\frac{\partial^2 u}{\partial t^2}(x,t) - \triangle u(x,t) \right) \mathrm{d}^{\mathrm{n}} x = 0. \end{aligned}$$

Here we applied once the divergence theorem to the vector field $\frac{\partial u}{\partial t} \nabla u$ which vanishes at $\partial\Omega \times [0,T]$ together with u and $\frac{\partial u}{\partial t}$. Initially the energy is zero $e(0) = 0$. Since the energy is non negative it stays zero for all positive times $t > 0$. This shows that u is constant and vanishes on $\Omega \times [0,T)$ since it vanishes initially. □

The proof gives the same conclusion if we assume that the normal derivative $\nabla u(x,t) \cdot N(x,t)$ is given on $\partial\Omega \times [0,T]$ instead of the values of $u(x,t)$.

Finally we give a simple proof that the length of the speed of propagation is bounded by 1.

Theorem 5.4. *If u is any solution of the homogeneous wave equation obeying $u = \frac{\partial u}{\partial t} = 0$ on $B(x_0, t_0)$ for $t = 0$, then u vanishes on the cone $\{(x,t) \mid |x - x_0| \leq t_0 - t, t > 0\}$.*

Proof. Again we calculate the time derivative of the energy

$$
\begin{aligned}
e(t) &= \frac{1}{2}\int_{B(x_0,t_0-t)} \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + (\nabla u(x,t))^2\right) \mathrm{d}^{\mathrm{n}}x \quad \text{as} \\
\frac{de}{dt}(t) &= \frac{1}{2}\frac{d}{dt}\int_0^{t_0-t}\int_{\partial B(x_0,s)} \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + (\nabla u(x,t))^2\right) \mathrm{d}\sigma(x)\,\mathrm{d}s \\
&= \int_{B(x_0,t_0-t)} \left(\frac{\partial^2 u}{\partial t^2}(x,t)\frac{\partial u}{\partial t}(x,t) + \frac{\partial \nabla u}{\partial t}(x,t)\nabla u(x,t)\right) \mathrm{d}^{\mathrm{n}}x \\
&\quad - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + (\nabla u(x,t))^2\right) \mathrm{d}\sigma(x) \\
&= \int_{B(x_0,t_0-t)} \frac{\partial u}{\partial t}(x,t)\left(\frac{\partial^2 u}{\partial t^2}(x,t) - \triangle u(x,t)\right) \mathrm{d}^{\mathrm{n}}x \\
&\quad + \int_{\partial B(x_0,t_0-t)} \left(\frac{\partial u}{\partial t}(x,t)\nabla u(x,t)\cdot N(x,t) - \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 - \frac{1}{2}(\nabla u(x,t))^2\right) \mathrm{d}\sigma(x) \\
&= \int_{\partial B(x_0,t_0-t)} \left(\frac{\partial u}{\partial t}(x,t)\nabla u(x,t)\cdot N(x,t) - \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 - \frac{1}{2}(\nabla u(x,t))^2\right) \mathrm{d}\sigma(x).
\end{aligned}
$$

Since the outer normal has length one we derive

$$\frac{\partial u}{\partial t}(x,t)\nabla u(x,t)\cdot N(x,t) \le \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \frac{1}{2}(\nabla u(x,t))^2$$

with $a = \nabla u(x,t)$ and $b = \frac{\partial u}{\partial t}(x,t)N(x,t)$ from the following inequality:

$$a\cdot b \le a\cdot b + \frac{1}{2}(a-b)\cdot(a-b) = \frac{1}{2}a^2 + \frac{1}{2}b^2. \tag{5.1}$$

So by $\dot{e}(t) \le 0$ the energy is monotonically decreasing. Because the energy is non-negative and vanishes initially it stays zero for all positive times in $t \in [0, t_0]$. This implies $u = 0$ on $\{(x,t) \mid |x - x_0| \le t_0 - t, t > 0\}$. □

By the invariance with respect to time reversal we can also deduce the vanishing of u on the cone $\{(x,t) \mid |x - x_0| < t_0 + t, t < 0\}$ from the vanishing of u and $\frac{\partial u}{\partial t} = 0$ on $(x,t) \in B(x_0, t_0) \times \{0\}$.

Finally, let us briefly tour the energy method for the Laplace and heat equations. Suppose that we have a solution u to the equation $-\triangle u = 0$. Consider what happens if we multiply this by u and integrate:

$$0 = \int_\Omega -\triangle u\, u\, \mathrm{d}^{\mathrm{n}}x = \int_\Omega |\nabla u|^2\, \mathrm{d}^{\mathrm{n}}x.$$

By the fundamental lemma of the calculus of variations, we know that the integrand is zero, i.e. u is constant. If we further know that u is zero on the boundary, then we have that $u \equiv 0$.

In analogy to the Laplace equation one can show the uniqueness for the heat equation. From intuition, the energy of a solution to the heat equation should be proportional to the total temperature, which we know is conserved. However we have seen that it is important to have positive functions, so that we can conclude that the function is zero if the integral is. Therefore we look at a simple positive quantity and define

$$e(t) = \int_\Omega u^2(x,t)\,\mathrm{d}^{\mathrm{n}}x.$$

If u solves the homogeneous heat equation and vanishes at the boundary of Ω, then this functional is monotonically decreasing with respect to time:

$$\dot{e}(t) = 2\int_\Omega u(x,t)\dot{u}(x,t)\,\mathrm{d}^{\mathrm{n}}x = 2\int_\Omega u(x,t)\triangle u(x,t)\,\mathrm{d}^{\mathrm{n}}x = -2\int_\Omega |\nabla u(x,t)|^2\,\mathrm{d}^{\mathrm{n}}x \le 0.$$

If $u(x,t)$ vanishes at $t = 0$, and if $u(\cdot,t)$ and $\nabla u(\cdot,t)$ are square integrable for $t > 0$, then u vanishes identically since $\nabla u(\cdot,t)$ vanishes and $u(\cdot,t)$ is constant for $t > 0$.

This idea is strong enough to show the uniqueness of the solution of the Dirichlet problem for all three of the second order equations in this course. In fact, Dirichlet's insight was that the unique solution of Dirichlet's Problem solves the following variational problem:

Dirichlet's Principle 5.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded and open and obey the assumptions of the Divergence Theorem. For continuous real functions f on $\bar{\Omega}$ and g on $\partial\Omega$ the solution u of the Dirichlet Problem 3.14 is the minimizer of the following functional:*

$$I : \{w \in C^2(\bar{\Omega}) \mid w|_{\partial\Omega} = g\} \to \mathbb{R}, \qquad w \mapsto I(w) = \int_\Omega \left(\frac{1}{2}\nabla w\cdot\nabla w - wf\right)\mathrm{d}^{\mathrm{n}}x.$$

Proof. Let u be a solution of the Dirichlet Problem and w another function in the domain $\{w \in C^2(\bar{\Omega}) \mid w|_{\partial\Omega} = g\}$ of I. An integration by parts yields

$$0 = \int_\Omega (-\triangle u - f)(u - w)\,\mathrm{d}^{\mathrm{n}}x = \int_\Omega (\nabla u\cdot\nabla(u - w) - f(u - w))\,\mathrm{d}^{\mathrm{n}}x.$$

We can rearrange this to give two expressions that are almost $I(u)$ and $I(w)$. All that remains is to deal with the mixed term $\nabla u\cdot\nabla w$ using the Cauchy-Schwarz inequality (5.1).

$$\begin{aligned}\int_\Omega (\nabla u\cdot\nabla u - fu)\,\mathrm{d}^{\mathrm{n}}x &= \int_\Omega (\nabla u\cdot\nabla w - fw)\,\mathrm{d}^{\mathrm{n}}x \le \\ &\le \int_\Omega \frac{1}{2}\nabla u\cdot\nabla u\,\mathrm{d}^{\mathrm{n}}x + \int_\Omega\left(\frac{1}{2}\nabla w\cdot\nabla w - fw\right)\mathrm{d}^{\mathrm{n}}x\end{aligned}$$

Moving the term across shows $I(u) \leq I(w)$.

If, conversely, u is a minimum, then all $v \in C^2(\bar{\Omega})$ which vanish on $\partial\Omega$ obey

$$0 = \frac{d}{dt} I(u+tv)\Big|_{t=0} = \frac{d}{dt}\left(I(u) + t\int_\Omega (\nabla u \cdot \nabla v - fv)\,\mathrm{d}^{\mathrm{n}}x + \frac{t^2}{2}\int_\Omega \nabla v \cdot \nabla v\,\mathrm{d}^{\mathrm{n}}x\right)\Big|_{t=0}$$
$$= \int_\Omega (\nabla u \cdot \nabla v - fv)\,\mathrm{d}^{\mathrm{n}}x = \int_\Omega (-\triangle u - f)v\,\mathrm{d}^{\mathrm{n}}x.$$

The final integration by parts shows $-\triangle u = f$ on Ω. □

This result naturally suggests a way of showing the existence of a solution to the Dirichlet problem for the Poisson equation, namely to show that there is a function that achieves this minimum. Dirichlet offered this as a proof of the existence, however he took for granted the existence of a function that attains the minimum.

The validity of Dirichlet's proof was questioned by Weierstrass, who offered the following example of a functional that had an infimum but no minimum. We use this as an example of the subtlety that exists, even though it does not directly involve the Laplace equation. Consider the set of functions in $C^1([-1,1])$ with the boundary condition $\varphi(-1) = -1$ and $\varphi(-1) = 1$. On this set, consider the functional $J(\varphi) = \int_{-1}^{1} (x\phi')^2\,\mathrm{d}x$. Clearly we have $J \geq 0$. On the other hand, consider the following family of functions

$$\varphi_\varepsilon(x) = \begin{cases} -1 & \text{if } x < -\varepsilon, \\ -1 + \varepsilon^{-1}(x+1) & \text{if } |x| < \varepsilon, \\ 1 & \text{if } x > \varepsilon \end{cases}$$

Strictly speaking, these function are not continuously differentiable, so we should smooth the function at $x = \pm\varepsilon$. This will not affect the following calculation significantly. We compute

$$J(\varphi_\varepsilon) = \int_{-\varepsilon}^{\varepsilon} (x\varepsilon^{-1})^2\,\mathrm{d}x = \varepsilon^{-2}\frac{1}{3}x^3\Big|_{-\varepsilon}^{\varepsilon} = \frac{2}{3}\varepsilon.$$

This shows that the infimum of J is exactly 0. However there is no function that achieves this value. If $J(\varphi) = 0$ then we must have $x\varphi' = 0$. This forces $\varphi'(x) = 0$ for $x \in [-1,1] \setminus \{0\}$ and hence for all x by continuity. But there is no constant function φ that meets the boundary condition.

In order that we finish the course on a positive note, we will consider one final example. This deals with some of the subtleties of Dirichlet's principle, which also also goes by the name "the direct method of the calculus of variations". In the previous example, the issue is that limit of φ_ε is not a continuous function (let alone a C^1 function). This is typical and one must choose a larger class of functions, usually a Sobolev space. In this example we define a notion of weak solution of the Laplace equation, namely we say that

$u \in \{w \in C(\bar{\Omega}) \mid w|_{\partial\Omega} = g\}$ on a bounded domain Ω is a *weak solution* of the Laplace equation if for any test function $\varphi \in C_0^\infty(\Omega)$ it obeys

$$\int_\Omega u \triangle \varphi \, \mathrm{d}^\mathrm{n} x = 0.$$

This is precisely the statement that the distribution F_u is harmonic, which explains the use of the term 'weak', but we will not use anything from Chapter 3. By integration by parts, a (twice continuously differentiable) harmonic function is a weak solution.

Suppose that we have a sequence of weak solutions u_k that converge locally uniformly to a function u. We can show that u is also a weak solution. By local uniform convergence, we mean that every point in Ω has a neighbourhood V such that $\|u - u_k\|_{L^\infty(V)}$ tends to zero. Equivalently we can say that the restrictions $u_k|_V$ converge uniformly to $u|_V$. This latter phrasing shows that u is continuous, since it is the uniform limit of continuous functions. Clearly $u|_{\partial\Omega} = g$. It remains to show that u has the necessary integral property.

$$\begin{aligned}\left|\int_\Omega u \triangle \varphi \, \mathrm{d}^\mathrm{n} x\right| &= \left|\int_\Omega u \triangle \varphi \, \mathrm{d}^\mathrm{n} x - \lim_{k\to\infty} \int_\Omega u_k \triangle \varphi \, \mathrm{d}^\mathrm{n} x\right| = \lim_{k\to\infty} \left|\int_\Omega (u - u_k) \triangle \varphi \, \mathrm{d}^\mathrm{n} x\right| \\ &\le \lim_{k\to\infty} \int_\Omega |u - u_k| \, |\triangle \varphi| \, \mathrm{d}^\mathrm{n} x = \lim_{k\to\infty} \int_{\operatorname{supp}\varphi} |u - u_k| \, |\triangle \varphi| \, \mathrm{d}^\mathrm{n} x.\end{aligned}$$

The point of the last step, reducing the integration to the support of φ, is that the support is compact. For each point of the support, choose a neighbourhood V such that we have uniform convergence. Because of compactness, we can choose finitely many V_i such that the support remains covered. Moreover the integrand is positive, so if we integrate certain parts of the domain multiple times, it only increases the value. Thus we continue with our calculation

$$\le \lim_{k\to\infty} \sum_i \int_{V_i} |u - u_k| \, |\triangle \varphi| \, \mathrm{d}^\mathrm{n} x \le \lim_{k\to\infty} \sum_i \|u - u_k\|_{L^\infty(V_i)} \int_{V_i} |\triangle \varphi| \, \mathrm{d}^\mathrm{n} x \to 0.$$

This is only possible if $\int_\Omega u \triangle \varphi \, \mathrm{d}^\mathrm{n} x = 0$. I hope that this gives you an idea of what goes into an existence proof using the energy method.

Chapter 6

Literature

This script derives from the script of Prof Martin Schmidt and I am very thankful to have had such a strong base from which to work.

In PDEs there are two gospels: Evans' "Partial Differential Equations" and Gilbarg-Trudinger's "Elliptic Partial Differential Equations of Second Order". The later is focused on general elliptic theory, which is not the focus in this course. So for the student who want to dig deeper I would recommend Evans.

Overall, our course is most similar to Han's "A Basic Course in Partial Differential Equations". There are also a number of other sources that might be a useful supplement for particular sections of the script. We have mentioned Lars Hörmander's "Linear Partial Differential Operators" with respect to distributions. I also drew from Eskin's "Lectures on Linear Partial Differential Equations" for the material on Fourier analysis. As you might have deduced, there are a great many textbooks that cover the material and many of them are good; find one that speaks to you.